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Research Article

End-Completely-Regular and End-Inverse Lexicographic Products of Graphs

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A graph X is said to be End-completely-regular (resp., End-inverse) if its endomorphism monoid $\mathrm{End}(X)$ is completely regular (resp., inverse). In this paper, we will show that if $X[Y]$ is End-completely-regular (resp., End-inverse), then both X and Y are End-completely-regular (resp., End-inverse). We give several approaches to construct new End-completely-regular graphs by means of the lexicographic products of two graphs with certain conditions. In particular, we determine the End-completely-regular and End-inverse lexicographic products of bipartite graphs.

1. Introduction and Preliminary Concepts

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years, much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The aim of this research is to develop further relationship between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. The bipartite graph is a class of famous graphs. Their endomorphism monoids are studied by several authors. In [1], the connected bipartite graphs whose endomorphism monoids are regular were explicitly found. In [2], Fan gave a characterization of connected bipartite graphs with an orthodox monoid. The joins of bipartite graphs with regular endomorphism monoids were characterized in [3]. The joins of bipartite graphs with completely regular endomorphism monoids were characterized in [4]. The endomorphism monoids and endomorphism regularity of graphs were considered by several authors (see [5–8]). In this paper, we will characterize the End-completely-regular and End-inverse lexicographic products of two graphs. We give several approaches to construct new End-completely-regular graphs by means of the lexicographic products of two graphs with certain conditions. In particular, we will determine the End-completely-regular and End-inverse lexicographic products of bipartite graphs.

The graphs X considered in this paper are undirected finite simple graphs. The vertex set of X is denoted by $V(X)$ and the edge set of X is denoted by $E(X)$. If two vertices x_1 and x_2 are adjacent in X, the edge connecting x_1 and x_2 is denoted by $\{x_1, x_2\}$ and we write $\{x_1, x_2\} \in E(X)$. A subgraph H is called an *induced subgraph* of X if for any $a, b \in V(H)$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(X)$. A graph X is called *bipartite* if X has no odd cycle. It is known that if a graph X is a bipartite graph, then its vertex set can be partitioned into two disjoint nonempty subsets such that no edge joins two vertices in the same set.

Let X and Y be two graphs. The *join* of X and Y, denoted by $X + Y$, is a graph such that $V(X + Y) = V(X) \cup V(Y)$ and $E(X + Y) = E(X) \cup E(Y) \cup \{\{x_1, x_2\} \mid x_1 \in V(X), x_2 \in V(Y)\}$. The *lexicographic product* of X and Y, denoted by $X[Y]$, is a graph with vertex set $V(X[Y]) = V(X) \times V(Y)$, and with edge set $E(X[Y]) = \{\{(x, y), (x_1, y_1)\} \mid \{x, x_1\} \in E(X)$, or $x = x_1$ and $\{y, y_1\} \in E(Y)\}$. Denote $Y_x = \{(x, y) \mid y \in V(Y)\}$ for any $x \in V(X)$.

Let X and Y be graphs. A mapping f from $V(X)$ to $V(Y)$ is called a *homomorphism* (from X to Y) if $\{x_1, x_2\} \in E(X)$ implies that $\{f(x_1), f(x_2)\} \in E(Y)$. A homomorphism f is called an *isomorphism* if f is bijective and f^{-1} is a homomorphism. A homomorphism (resp., isomorphism) f from X to itself is called an *endomorphism* (resp., *automorphism*) of X (see [9]). The sets of all endomorphisms and automorphisms

of X are denoted by $\mathrm{End}(X)$ and $\mathrm{Aut}(X)$, respectively. A graph X is said to be *unretractive* if $\mathrm{End}(X) = \mathrm{Aut}(X)$. For any $f \in \mathrm{End}(X)$, it is easy to see that $f \in \mathrm{Aut}(X)$ if and only if f is injective.

A *retraction* of a graph X is a homomorphism f from X to a subgraph Y of X such that the restriction $f|_Y$ of f to $V(Y)$ is the identity mapping on $V(Y)$. In this case, Y is called a *retract* of X. It is known that the idempotents of $\mathrm{End}(X)$ are retractions of X. Denote by $\mathrm{Idpt}(X)$ the set of all idempotents of $\mathrm{End}(X)$. Let f be an endomorphism of a graph X. A subgraph of X is called the *endomorphic image* of X under f, denoted by I_f, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(X)$. By ρ_f we denote the equivalence relation on $V(X)$ induced by f; that is, for $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class containing $a \in V(X)$ with respect to ρ_f.

An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. An element a of a semigroup S is called *completely regular* if $a = axa$ and $xa = ax$ hold for some $x \in S$. A semigroup S is called *regular* (resp., *completely regular*) if all its elements are regular (resp., completely regular). An *inverse semigroup* is a regular semigroup in which the idempotents commute. A graph X is said to be End-regular (resp., End-completely-regular, End-inverse) if its endomorphism monoid $\mathrm{End}(X)$ is regular (resp., completely regular, inverse). Clearly, End-completely-regular graphs as well as End-inverse graphs are End-regular.

For undefined notation and terminology in this paper, the reader is referred to [9–14]. We list some known results which will be used in the sequel.

Lemma 1 (see [2]). *If $X[Y]$ is End-regular, then both X and Y are End-regular.*

Lemma 2 (see [15]). *Let G be a graph and let $f \in End(G)$. Then f is completely regular if and only if $f|_{I_f} \in Aut(I_f)$.*

Lemma 3 (see [15]). *Let X be a bipartite graph. Then X is End-completely-regular if and only if X is one of K_1, K_2, P_2, $2K_1$, $2K_2$, and $K_1 \bigcup K_2$.*

Lemma 4 (see [4]). *Let X and Y be two bipartite graphs. Then $X + Y$ is End-completely-regular if and only if one of them is End-completely-regular and the other is K_1 or K_2.*

Lemma 5 (see [16]). *Let G be a graph and $f \in End(G)$. Then f is completely regular if and only if there exists $g \in Idpt(G)$ such that $\rho_g = \rho_f$ and $I_g = I_f$.*

Lemma 6 (see [4]). *Let X be a bipartite graph. Then X is End-inverse if and only if $X = K_1$ or $X = K_2$.*

Lemma 7 (see [17]). *Let X and Y be two graphs. Then $End(X[Y]) = End(X)[End(Y)]$ if and only if for any $f \in End(X[Y])$ and $x \in V(X)$, there exists $x' \in V(X)$ such that $f(Y_x) \subseteq Y_{x'}$.*

2. Main Results

In this section, we will characterize the End-completely-regular and End-inverse lexicographic products of two graphs. We first show that if $X[Y]$ is End-completely-regular, then both X and Y are End-completely-regular.

Theorem 8. *Let X and Y be two graphs. If $X[Y]$ is End-completely-regular, then both X and Y are End-completely-regular.*

Proof. By Lemma 2, to show that X is End-completely-regular, it is only necessary to verify that $f|_{I_f}$ is an automorphism of I_f for each $f \in \mathrm{End}(X)$. Define a mapping F from $V(X[Y])$ to itself by

$$F((x, y)) = (f(x), y) \quad \forall (x, y) \in V(X[Y]). \tag{1}$$

Then $F \in \mathrm{End}(X[Y])$. Since $X[Y]$ is End-completely-regular, by Lemma 2, $F|_{I_F}$ is an automorphism of I_F. It is easy to see that $I_F = I_f[Y]$. For any distinct $x_1, x_2 \in V(I_f)$ and $y \in V(Y)$, $F((x_1, y)) = (f(x_1), y)$ and $F((x_2, y)) = (f(x_2), y)$ hold. Since $F|_{I_F}$ is an automorphism of I_F, $(f(x_1), y) \neq (f(x_2), y)$. Hence $f(x_1) \neq f(x_2)$ and so $f|_{I_f}$ is an automorphism of I_f.

Let $g \in \mathrm{End}(Y)$. Define a mapping G from $V(X[Y])$ to itself by

$$G((x, y)) = (x, g(y)) \quad \forall (x, y) \in V(X[Y]). \tag{2}$$

Then $G \in \mathrm{End}(X[Y])$. Since $X[Y]$ is End-completely-regular, by Lemma 2, $G|_{I_G}$ is an automorphism of I_G. It is easy to see that $I_G = X[I_g]$. For any $x \in V(X)$ and $y_1, y_2 \in V(I_g)$, $G((x, y_1)) = (x, g(y_1))$ and $G((x, y_2)) = (x, g(y_2))$. Since $G|_{I_G}$ is an automorphism of I_G, $(x, g(y_1)) \neq (x, g(y_2))$, we get that $g(y_1) \neq g(y_2)$ and so $g|_{I_g}$ is an automorphism of I_g, as required. □

The following example shows that X and Y being End-completely-regular does not yield that $X[Y]$ is End-completely-regular.

Example 9. Let X and Y be two graphs with $V(X) = \{x_1, x_2\}$, $V(Y) = \{y_1, y_2\}$, $E(X) = \{\{x_1, x_2\}\}$, and $E(Y) = \phi$. By Lemma 3, X and Y are End-completely-regular. It is easy to see that $X[Y] \cong C_4$. Also by Lemma 3, this is not End-completely-regular.

In the following, we give some sufficient conditions for $X[Y]$ to be End-completely-regular. To this aim, we need the following result due to Fan [17].

Lemma 10 (see [17]). *Let X and Y be two K_3-free connected graphs. If $girth(X)$ or $girth(Y)$ is odd, then $End(X[Y]) = End(X)[End(Y)]$, where $End(X)[End(Y)]$ is the wreath product of the monoids $End(X)$ and $End(Y)$.*

Let X and Y be two K_3-free connected graphs such that $\mathrm{girth}(X)$ or $\mathrm{girth}(Y)$ is odd. In [2], Fan proved that if both of X and Y are End-regular and one of them is unretractive, then $X[Y]$ is End-regular. Here we prove that if X is an End-completely-regular graph and Y is an unretractive graph, then $X[Y]$ is End-completely-regular.

Theorem 11. *Let X and Y be two K_3-free connected graphs with girth(X) or girth(Y) being odd, and assume that*

(1) *X is End-completely-regular,*

(2) *Y is unretractive.*

Then $X[Y]$ is End-completely-regular.

Proof. Let X and Y be two graphs satisfying the assumptions. To show that $X[Y]$ is End-completely-regular, we prove that for any $F \in \mathrm{End}(X[Y])$, there exists an idempotent endomorphism $G \in \mathrm{End}(X[Y])$ such that $\rho_F = \rho_G$ and $I_F = I_G$.

Let $F \in \mathrm{End}(X)[\mathrm{End}(Y)]$. Since $\mathrm{End}(X[Y]) = \mathrm{End}(X)[\mathrm{End}(Y)]$, $F = (s, f)$ for some $s \in \mathrm{End}(X)$ and $f \in \mathrm{End}(Y)^{V(X)}$. Thus, for any $u \in V(X)$, there exists $f_u = f(u) \in \mathrm{End}(Y)$. Let X and Y be K_3-free connected graphs with girth(X) or girth(Y) being odd. By Lemma 7, for any $u \in V(X)$, $F(Y_u) \subseteq Y_v$ for some $v \in V(X)$. Note that Y is unretractive. Then $F(Y_u) = Y_v$. Since X is End-completely-regular and $s \in \mathrm{End}(X)$, by Lemma 5, there exists $t \in \mathrm{Idpt}(X)$ such that $\rho_t = \rho_s$ and $I_t = I_s$. Clearly, I_s is an induced subgraph of X. Hence $I_F = I_s[Y]$ is an induced subgraph of $X[Y]$.

Since X is End-completely-regular, $s|_{I_s}$ is an automorphism of I_s. Thus for any $u \in I_s$, there exists only one vertex $u_1 \in I_s$ such that $s(u_1) = u$. Then $F(Y_{u_1}) = Y_u$. Now for any $(u, v) \in I_F$, there exists only one vertex $(u_1, v_1) \in I_F$ such that $F((u_1, v_1)) = (u, v)$. Define a mapping G from $V(X[Y])$ to itself in the following way. If $(x, y) \in V(I_F)$, then $G((x, y)) = (x, y)$; if $(x, y) \notin V(I_F)$, then $F((x, y)) = (u, v)$ for some $(u, v) \in V(I_F)$. Now let $G((x, y)) = (u_1, v_1)$, where (u_1, v_1) is the only vertex in $V(I_F)$ such that $F((u_1, v_1)) = (u, v)$. Then it is easy to see that G is well-defined. Let $(x, y) \in V(X[Y])$. If $(x, y) \in V(I_F)$, then $x \in I_s$. Thus $t(x) = x$. Hence $G((x, y)) = (x, y) \in Y_{t(x)}$. If $(x, y) \notin V(I_F)$, then $t(x) = u_1$. Hence $G((x, y)) = (u_1, v_1) \in Y_{t(x)}$. Therefore, $G((x, y)) \in Y_{t(x)}$ for any $(x, y) \in V(X[Y])$.

Let $(x_1, y_1), (x_2, y_2) \in V(X[Y])$ be such that $\{(x_1, y_1), (x_2, y_2)\} \in E(X[Y])$. If $(x_1, y_1), (x_2, y_2) \in V(I_F)$, then $\{G((x_1, y_1)), G((x_2, y_2))\} = \{(x_1, y_1), (x_2, y_2)\} \in E(X[Y])$. If $(x_1, y_1) \in V(I_F)$ and $(x_2, y_2) \notin V(I_F)$, then $x_1 \neq x_2$ and $\{x_1, x_2\} \in E(X)$. Thus $G((x_1, y_1)) \in Y_{t(x_1)}$ and $G((x_2, y_2)) \in Y_{t(x_2)}$. Since $t \in \mathrm{Idpt}(X)$ and $\{x_1, x_2\} \in E(X)$, $\{t(x_1), t(x_2)\} \in E(X)$. Hence $\{G((x_1, y_1)), G((x_2, y_2))\} \in E(X[Y])$. If $(x_1, y_1) \notin V(I_F)$ and $(x_2, y_2) \notin V(I_F)$, there are two cases.

Case 1. Assume that $\{x_1, x_2\} \in E(X)$. Then $G((x_1, y_1)) \in Y_{t(x_1)}$ and $G((x_2, y_2)) \in Y_{t(x_2)}$. Since $t \in \mathrm{Idpt}(X)$ and $\{x_1, x_2\} \in E(X)$, $\{t(x_1), t(x_2)\} \in E(X)$. Hence we have $\{G((x_1, y_1)), G((x_2, y_2))\} \in E(X[Y])$.

Case 2. Assume that $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$. Then we have $(x_1, y_1), (x_2, y_2) \in Y_{x_1}$ and $G((x_1, y_1)), G((x_2, y_2)) \in Y_{t(x_1)}$. Since $G|_{Y_{x_1}}$ is an isomorphism from Y_{x_1} to $Y_{t(x_1)}$, $\{G((x_1, y_1)), G((x_2, y_2))\} \in E(X[Y])$. Therefore, $G \in \mathrm{End}(X[Y])$.

If $(x, y) \in V(I_F)$, then $G^2((x, y)) = G((x, y)) = (x, y)$. If $(x, y) \notin V(I_F)$, then $G((x, y)) \in V(I_F)$. Thus $G^2((x, y)) = G(G((x, y))) = G((x, y))$. Hence $G \in \mathrm{Idpt}(X[Y])$. Clearly, $I_G = I_F$.

Suppose $[(x_1, y_1)]_{\rho_F} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$ for some $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \in V(X[Y])$. In fact, it is easy to prove that $\rho_F \subseteq \rho_G \subseteq \rho_F$. Let $(x_1, y_1)\rho_F(x_2, y_2)$. Then, by the definition of G, we have $G((x_1, y_1)) = G((x_2, y_2)) = (u_1, v_1)$ for some $(u_1, v_1) \in V(I_F)$ with $F((x_1, y_1)) = F(u_1, v_1) = F((x_2, y_2))$. So $(x_1, y_1)\rho_G(x_2, y_2)$ and thus $\rho_F \subseteq \rho_G \subseteq \rho_F$. Hence $\rho_F = \rho_G$. □

Next we start to seek the conditions for bipartite graphs X and Y under which $X[Y]$ is End-completely-regular.

Lemma 12. *Let X be a graph and R be a retract of X. If $R[Y]$ is not End-completely-regular, then $X[Y]$ is not End-completely-regular.*

Proof. Let R be a retract of X. Then there exists $f \in \mathrm{Idpt}(X)$ such that $I_f = R$. Let $g \in \mathrm{End}(R[Y])$. Since $R[Y]$ is not End-completely-regular, there exists $g \in \mathrm{End}(R[Y])$ such that g is not completely regular. By Lemma 2, $g|_{I_g}$ is not an automorphism of I_g. Thus there exist $x_1, x_2 \in I_g$ with $x_1 \neq x_2$ such that $g(x_1) = g(x_2)$. Define a mapping F from $V(X[Y])$ to itself by

$$F((x, y)) = (f(x), y) \quad \forall (x, y) \in V(X[Y]). \tag{3}$$

Then $F \in \mathrm{End}(X[Y])$ and $I_F = R[Y]$. Now it is easy to see that $gF \in \mathrm{End}(X[Y])$ and $I_{gF} = I_g$. It follows from $(gF)(x_1) = (gF)(x_2)$ that $(gF)|_{I_{gF}}$ is not an automorphism of I_{gF}. Hence $X[Y]$ is not End-completely-regular. □

Lemma 13. *Let X and Y be two graphs. If at least one of X and Y is not End-completely-regular, then $X \cup Y$ is not End-completely-regular (where $X \cup Y$ is the disjoint union of X and Y).*

Proof. Without loss of generality, we may suppose that X is not End-completely-regular. By Lemma 2, there exists $f \in \mathrm{End}(X)$ such that $f|_{I_f}$ is not an automorphism of I_f. Define a mapping F from $V(X \cup Y)$ to itself by

$$F(x) = \begin{cases} f(x), & x \in V(X), \\ x, & x \in V(Y). \end{cases} \tag{4}$$

Then $F \in \mathrm{End}(X \cup Y)$. Now it is easy to see that $I_F = I_f \cup Y$ and $F(x) = f(x)$ for any $x \in V(X)$. Since $f|_{I_f}$ is not an automorphism of I_f, $F|_{I_F}$ is not an automorphism of I_F. Hence $X \cup Y$ is not End-completely-regular. □

Theorem 14. *Let X and Y be two bipartite graphs. Then $X[Y]$ is End-completely-regular if and only if*

(1) *$X = K_1$ and Y is End-completely-regular or*

(2) *X is End-completely-regular and $Y = K_1$ or K_2.*

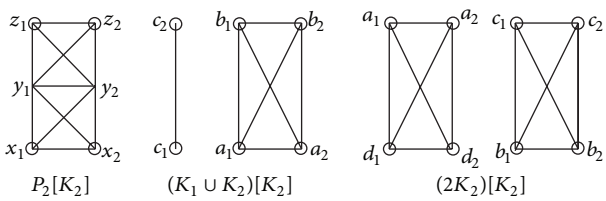

Figure 1: Graphs $P_2[K_2]$, $(K_1 \cup K_2)[K_2]$, and $(2K_2)[K_2]$.

Proof

Sufficiency. Since $K_1[Y] = Y$ and $X[K_1] = X$, we have immediately that $K_1[Y](X[K_1])$ is End-completely-regular if and only if $Y(X)$ is End-completely-regular. If $X = Y = K_2$, then $X[Y] = K_4$. Thus $\mathrm{End}(X[Y])$ is a group. Since any group is a completely regular semigroup, $X[Y]$ is End-completely-regular. If $X = 2K_1$ and $Y = K_2$, then $X[Y] = 2K_2$. By Lemma 3, it is End-completely-regular. In the following, we show that $X[Y]$ is End-completely-regular for the following cases (see Figure 1).

Case 1. $X = P_2$ and $Y = K_2$. Let $f \in \mathrm{End}(X[Y])$. If $[x_1]_{\rho_f} = \{x_1\}$, then $[x_2]_{\rho_f} = \{x_2\}$. Otherwise, $f(x_2) = f(z_1)$ or $f(x_2) = f(z_2)$. Without loss of generality, we can suppose $f(x_2) = f(z_1)$. Since z_1 is adjacent to every vertex of $\{z_2, y_1, y_2\}$ and $\{x_1, x_2\} \in E(X[Y])$, $f(z_1)$ is adjacent to every vertex of $\{f(z_2), f(y_1), f(y_2), f(x_1)\}$. Note that there is no vertex in $X[Y]$ adjacent to 4 vertices. This is a contradiction. Hence $f \in \mathrm{Aut}(X[Y])$ and so f is completely regular. If $[x_1]_{\rho_f} \neq \{x_1\}$, then $f(x_1) = f(z_1)$ or $f(x_1) = f(z_2)$. Without loss of generality, we can suppose $f(x_1) = f(z_1)$. Then $f(x_2) = f(z_2)$. Otherwise, a similar argument as above will show that $f(x_1)$ is adjacent to every vertex of $\{f(x_2), f(y_1), f(y_2), f(z_2)\}$, which yield a contradiction. Thus $I_f \cong K_4$. Since any endomorphism f maps a clique to a clique of the same size, $f(I_f) = I_f$. By Lemma 2, f is completely regular. Hence $P_2[K_2]$ is End-completely-regular.

Case 2. $X = K_1 \cup K_2$ and $Y = K_2$. Let $f \in \mathrm{End}(X[Y])$. If $[c_1]_{\rho_f} = \{c_1\}$, then $[c_2]_{\rho_f} = \{c_2\}$. Otherwise, $f(c_2) \in \{f(a_1), f(a_2), f(b_1), f(b_2)\}$. Without loss of generality, we can suppose $f(c_2) = f(a_1)$. Since any endomorphism f maps a clique to a clique of the same size and there is only one clique of size 4 in $X[Y]$, $\{f(a_1), f(a_2), f(b_1), f(b_2)\} = \{a_1, a_2, b_1, b_2\}$. Note that $\{c_1, c_2\} \in E(X[Y])$. Then $\{f(c_1), f(c_2)\} = \{f(c_1), f(a_1)\} \in E(X[Y])$. Thus $f(c_1) \in \{f(a_1), f(a_2), f(b_1), f(b_2)\}$, which is a contradiction. Clearly, $[x]_{\rho_f} = \{x\}$ for any $x \in \{a_1, a_2, b_1, b_2\}$. Hence $f \in \mathrm{Aut}(X[Y])$ and so f is completely regular. If $[c_1]_{\rho_f} \neq \{c_1\}$, then $f(c_1) = f(t)$ for some $t \in \{a_1, a_2, b_1, b_2\}$. Without loss of generality, we can suppose $f(c_1) = f(a_1)$. Then $f(c_2) \in \{f(a_2), f(b_1), f(b_2)\}$. Thus $I_f \cong K_4$. Hence $f(I_f) = I_f$ and f is completely regular. Consequently, $(K_1 \cup K_2)[K_2]$ is End-completely-regular.

Case 3. $X = 2K_2$ and $Y = K_2$. Let $f \in \mathrm{End}(X[Y])$. If $[x]_{\rho_f} = \{x\}$ for any $x \in V(X[Y])$, then $f \in \mathrm{Aut}(X[Y])$ and so f is completely regular. If $f(x) = f(y)$ for some $x, y \in V(X[Y])$ with $x \neq y$, without loss of generality, we can suppose $f(a_1) = f(c_1)$. Since b_1, b_2, c_1, c_2 is a clique of size 4 in $X[Y]$, $f(b_1)$, $f(b_2)$, $f(c_1)$, $f(c_2)$ is also a clique of size 4 in $X[Y]$. Note that a_2, d_1, d_2 are adjacent to a_1. Then $f(a_2)$, $f(d_1)$, $f(d_2)$ are adjacent to $f(a_1) = f(c_1)$. Thus $f(a_2), f(d_1), f(d_2) \in \{f(b_1), f(b_2), f(c_2)\}$ and $I_f \cong K_4$. Hence $f(I_f) = I_f$ and f is completely regular. Consequently, $(2K_2)[K_2]$ is End-completely-regular.

Necessity. We only need to show that $X[Y]$ is not End-completely-regular in the following cases.

Case 1 ($X = K_2$). Then $X[Y] = Y + Y$. By Lemma 4, $K_2[Y]$ is not End-completely-regular for the corresponding Y.

Case 2 ($X = P_2$). Then K_2 is a retract of X. Since $K_2[Y]$ is not End-completely-regular for $Y = P_2, 2K_1, 2K_2, K_1 \cup K_2$, by Lemma 12, $P_2[Y]$ is not End-completely-regular for the corresponding Y.

Case 3 ($X = 2K_1$). Then $X[Y] = 2Y$. If Y is bipartite, then $X[Y]$ is also bipartite. By Lemma 3, $(2K_1)[Y]$ is not End-completely-regular for the corresponding Y.

Case 4 ($X = 2K_2$). Then $X[Y] = 2(Y + Y)$. Since $Y + Y$ is not End-completely-regular for $Y = P_2, 2K_1, 2K_2, K_1 \cup K_2$, by Lemma 13, $(2K_2)[Y]$ is not End-completely-regular for the corresponding Y.

Case 5 ($X = K_1 \cup K_2$). Then $X[Y] = Y \cup (Y + Y)$. Since $Y + Y$ is not End-completely-regular for $Y = P_2, 2K_1, 2K_2, K_1 \cup K_2$, by Lemma 13, $(K_1 \cup K_2)[Y]$ is not End-completely-regular for the corresponding Y. □

Next we start to seek the conditions for a lexicographic product of bipartite graphs X and Y under which $X[Y]$ is End-inverse.

Theorem 15. *Let X and Y be two graphs. If $X[Y]$ is End-inverse, then both X and Y are End-inverse.*

Proof. Since $X[Y]$ is End-inverse, $X[Y]$ is End-regular. By Lemma 1, both X and Y are End-regular. To show that X is End-inverse, we only need to prove that the idempotents of $\mathrm{End}(X)$ commute.

Let f_1 and f_2 be two idempotents in $\mathrm{End}(X)$. Define two mappings g_1 and g_2 from $V(X[Y])$ to itself by

$$\begin{aligned} g_1((x,y)) &= (f_1(x), y) \quad \forall (x,y) \in V(X[Y]), \\ g_2((x,y)) &= (f_2(x), y) \quad \forall (x,y) \in V(X[Y]). \end{aligned} \tag{5}$$

Then g_1 and g_2 are two idempotents of $\mathrm{End}(X[Y])$ and so $g_1 g_2 = g_2 g_1$, since $X[Y]$ is End-inverse. For any $(x, y) \in V(X[Y])$, we have

$$\begin{aligned} (g_1 g_2)((x,y)) &= g_1((f_2(x), y)) = ((f_1 f_2)(x), y) \\ &= (g_2 g_1)((x,y)) = ((f_2 f_1)(x), y). \end{aligned} \tag{6}$$

Clearly, $f_1 f_2 = f_2 f_1$. Hence X is End-inverse.

Similarly, let f_3 and f_4 be two idempotents in End(Y). Define two mappings g_3 and g_4 from $V(X[Y])$ to itself by

$$\begin{aligned} g_3((x,y)) &= (x, f_3(y)) \quad \forall (x,y) \in V(X[Y]), \\ g_4((x,y)) &= (x, f_4(y)) \quad \forall (x,y) \in V(X[Y]). \end{aligned} \tag{7}$$

Then g_3 and g_4 are two idempotents of End($X[Y]$) and so $g_3 g_4 = g_4 g_3$, since $X[Y]$ is End-inverse. For any $(x, y) \in V(X[Y])$, we have

$$\begin{aligned} (g_3 g_4)((x,y)) &= (x, (f_3 f_4)(y)) = (g_4 g_3)((x,y)) \\ &= (x, (f_4 f_3)(y)). \end{aligned} \tag{8}$$

Clearly, $f_3 f_4 = f_4 f_3$. Hence Y is End-inverse, as required. □

The next theorem characterizes the End-inverse lexicographic products of bipartite graphs.

Theorem 16. *Let X and Y be two bipartite graphs. Then $X[Y]$ is End-inverse if and only if $X[Y]$ is one of $K_1[K_1]$, $K_1[K_2]$, $K_2[K_1]$, and $K_2[K_2]$.*

Proof

Necessity. This follows directly from Lemma 6 and Theorem 15.

Sufficiency. It is easy to see that $K_1[K_1] = K_1$, $K_1[K_2] = K_2[K_1] = K_2$, and $K_2[K_2] = K_4$ are End-inverse, since they are unretractive. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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