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Research Article

On Rationality of Kneading Determinants

Sheng Chen and Chao Xia

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

Correspondence should be addressed to Sheng Chen; schenhit@gmail.com

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In this section, R denotes a ring with identity I. We study conditions under which $I - A\lambda$ and $I - B\lambda$ are left coprime or right coprime, where $A, B \in R$. As applications, we get sufficient conditions under which the Kneading determinant of a finite rank pair of operators on an infinite dimensional space is rational.

1. Introduction

If A is a $k \times k$ matrix with rational coefficients, one has the well-known identity between formal power series:

$$\exp \sum_{n\geq 1} - \frac{\operatorname{tr}\left(A^n\right)}{n} z^n = \det\left(I - zA\right), \tag{1}$$

where $\operatorname{tr}(A^n)$ denotes the trace of matrix A^n (matrix A raised to the nth power) and I denotes the $k \times k$ identity matrix. This identity plays a significant role in the discussion of an important problem in dynamical systems theory. For more details see [1, 2].

We denote by $\mathscr{H}$ the infinite dimension vector space over $\mathbb{Q}$; the space of linear forms on $\mathscr{H}$ will be denoted, as usual, by $\mathscr{H}^*$, and the space of all linear endomorphisms on $\mathscr{H}$ will be denoted by $L(\mathscr{H})$. If $\psi \in L(\mathscr{H})$ and n is a nonnegative integer, the nth iterate ψ^n is defined recursively by $\psi^0 = \mathrm{Id}_{\mathscr{H}} \in L(\mathscr{H})$, $\psi^n = \psi \circ \psi^{n-1}$, for $n \geq 1$.

The subspace of $L(\mathscr{H})$ whose elements are the linear endomorphism on $\mathscr{H}$ with finite rank will be denoted by $L_{\mathrm{FR}}(\mathscr{H})$.

Let q be a positive integer; we use the symbol $\overline{h}$ to denote an element of $\mathscr{H}^q = \mathscr{H} \times \mathscr{H} \times \mathscr{H} \times \cdots \times \mathscr{H}$ (q times) and the symbol $\overline{\alpha}$ to denote an element of $\mathscr{H}^{*q}$; that is

$$\overline{h} = \left(h_1, \ldots, h_q\right), \qquad \overline{\alpha} = \left(\alpha_1, \ldots, \alpha_q\right). \tag{2}$$

Given $\overline{h} \in \mathscr{H}^q$ and $\overline{\alpha} \in \mathscr{H}^{*q}$, we define the finite rank endomorphism $\overline{\alpha} \otimes \overline{h} \in L_{\mathrm{FR}}(\mathscr{H})$ and the matrix $\mathbf{M}(\overline{\alpha}, \overline{h}) \in \mathbb{Q}^{q\times q}$ by setting

$$\overline{\alpha} \otimes \overline{h} = \sum_{p=1}^{q} \alpha_p \otimes h_p, \tag{3}$$

with the usual notation

$$\alpha \in \mathscr{H}^*, h \in \mathscr{H} : (\alpha \otimes h)(x) = \alpha(x) u, \quad x \in \mathscr{H},$$

$$\mathbf{M}\left(\overline{\alpha}; \overline{h}\right) = \begin{pmatrix} \alpha_1(h_1) & \cdots & \alpha_1(h_q) \\ \vdots & \ddots & \vdots \\ \alpha_q(u_1) & \cdots & \alpha_q(h_q) \end{pmatrix}. \tag{4}$$

Definition 1 (see [3]). A pair $(\varphi, \psi) \in L(\mathscr{H}) \times L(\mathscr{H})$ is said to have finite rank if $\psi - \varphi \in L_{\mathrm{FR}}(\mathscr{H})$.

Notice that if a pair (φ, ψ) has finite rank, then the pair (φ^n, ψ^n) also has finite rank for all $n \geq 1$, and therefore the trace of $\varphi^n - \psi^n$ is defined.

Definition 2 (see [3]). For any pair $(\varphi, \psi) \in L(\mathscr{H}) \times L(\mathscr{H})$ with finite rank, we define the Kneading determinant of (φ, ψ) as the following invertible element of $\mathbb{Q}[[z]]$:

$$\Delta_{(\varphi,\psi)} = \exp \sum_{n\geq 1} \frac{\operatorname{tr}\left(\varphi^n - \psi^n\right)}{n} z^n. \tag{5}$$

Remark 3. Kneading determinant was first studied by Milnor and Thurston in [4].

Let $\mathbb{Q}[[z]]^{q\times q}$ be the ring of the $q \times q$ matrices whose entries lie in $\mathbb{Q}[[z]]$. If $\varphi \in L(\mathscr{H})$, $\overline{u} \in \mathscr{H}^q$, and $\overline{\alpha} \in \mathscr{H}^{*q}$, we define the matrix $\mathbf{M}_\varphi(\overline{\alpha};\overline{u}) \in \mathbb{Q}[[z]]^{q\times q}$ by

$$\mathbf{M}_\varphi(\overline{\alpha};\overline{u}) = \begin{pmatrix} \sum_{n\geq 0}\alpha_1\varphi^n(u_1)z^n & \cdots & \sum_{n\geq 0}\alpha_1\varphi^n(u_q)z^n \\ \vdots & \ddots & \vdots \\ \sum_{n\geq 0}\alpha_q\varphi^n(u_1)z^n & \cdots & \sum_{n\geq 0}\alpha_q\varphi^n(u_q)z^n \end{pmatrix}. \tag{6}$$

Lemma 4 (see [3]). *Let $(\varphi,\psi) \in L(\mathscr{H}) \times L(\mathscr{H})$, $\overline{u} \in \mathscr{H}^q$, and $\overline{\alpha} \in \mathscr{H}^{*q}$ such that $\psi - \varphi = \overline{\alpha}\otimes\overline{u}$. Denote by $\mathbf{I}$ the $q\times q$ identity matrix. Then,*

$$\Delta_{(\varphi,\psi)} = \det\left(\mathbf{I} - z\mathbf{M}_\varphi(\overline{\alpha};\overline{u})\right) \tag{7}$$

holds in $\mathbb{Q}[[z]]$.

In general, any power series can be the Kneading determinant of some pair (φ,ψ) with finite rank (see [3]). So it is interesting to study conditions under which the Kneading determinant is a rational power series.

The following is the main result of this paper.

Theorem 5. *If $I - \varphi\lambda$ and $I - \psi\lambda$ are left coprime or right coprime, and $h = \psi - \varphi$ with finite rank, then $\Delta_{(\varphi,\psi)}$ is a rational function.*

2. Coprimeness of $I-A\lambda$ and $I-B\lambda$

In this section, R denotes a ring with identity I. We discuss conditions under which $I - A\lambda$ and $I - B\lambda$ are left coprime or right coprime, where $A, B \in R$.

We say $I - A\lambda$ and $I - B\lambda$ are left coprime if there exist polynomials $X(\lambda), Y(\lambda) \in R[\lambda]$ such that

$$X(\lambda)(I - A\lambda) + Y(\lambda)(I - B\lambda) = I. \tag{8}$$

Proposition 6. *$I - A\lambda$ and $I - B\lambda$ are left coprime if and only if there exist $X_0, X_1, \ldots, X_m \in R$ such that*

$$X_mH + X_{m-1}HB + X_{m-2}HB^2 + \cdots + X_0HB^m + B^{m+1} = 0, \tag{9}$$

for $H = A - B$.

Proof. If $I - A\lambda$ and $I - B\lambda$ are left coprime, there exist $X(\lambda) = X_0 + X_1\lambda + \cdots + X_m\lambda^m$ and $Y(\lambda) = Y_0 + Y_1\lambda + \cdots + Y_s\lambda^s$ such that

$$X(\lambda)(I - A\lambda) + Y(\lambda)(I - B\lambda) = I. \tag{10}$$

We can assume that s equals m. And we have

$$\begin{gathered} X_0 + Y_0 = I, \\ X_iA + Y_iB - X_{i+1} - Y_{i+1} = 0, \quad i = 1, 2, \ldots, m-1, \\ X_mA + Y_mB = 0. \end{gathered} \tag{11}$$

So,

$$Y_0 = I - X_0. \tag{12}$$

If we write $H = A - B$, then,

$$\begin{aligned} Y_1 &= -X_1 + X_0H + B, \\ Y_2 &= -X_2 + X_1H + X_0HB + B^2, \\ &\vdots \\ Y_m &= -X_m + X_{m-1}H + \cdots + X_0HB^{m-1} + B^m. \end{aligned} \tag{13}$$

So,

$$X_mH + X_{m-1}HB + X_{m-2}HB^2 + \cdots + X_0HB^m + B^{m+1} = 0. \tag{14}$$

Conversely, if there exist $X_0, X_1, \ldots, X_m \in R$ such that

$$X_mH + X_{m-1}HB + X_{m-2}HB^2 + \cdots + X_0HB^m + B^{m+1} = 0, \tag{15}$$

let

$$\begin{aligned} X(\lambda) &= X_0 + X_1\lambda + \cdots + X_m\lambda^m, \\ Y_0 &= I - X_0, \\ Y_1 &= -X_1 + X_0H + B, \\ Y_2 &= -X_2 + X_1H + X_0HB + B^2, \\ &\vdots \\ Y_m &= -X_m + X_{m-1}H + \cdots + X_0HB^{m-1} + B^m, \end{aligned} \tag{16}$$

and let

$$Y(\lambda) = Y_0 + Y_1\lambda + \cdots + Y_m\lambda^m. \tag{17}$$

Then,

$$X(\lambda)(I - A\lambda) + Y(\lambda)(I - B\lambda) = I. \tag{18}$$

□

Now we give the definition of right coprime. We say $I - A\lambda$ and $I - B\lambda$ are right coprime if there exist polynomials $X(\lambda), Y(\lambda) \in R[\lambda]$ such that

$$(I - A\lambda)X(\lambda) + (I - B\lambda)Y(\lambda) = I. \tag{19}$$

Proposition 7. *$I - A\lambda$ and $I - B\lambda$ are right coprime if and only if there exist $X_0, X_1, \ldots, X_s \in R$ such that*

$$HX_s + BHX_{s-1} + B^2HX_{s-2} + \cdots + B^sHX_0 + B^{s+1} = 0, \tag{20}$$

for $H = A - B$.

Proof. It is similar to the proof of Proposition 6. □

3. Proof of the Main Theorem

In this section, $\varphi, \psi \in L(\mathscr{H})$, where $\mathscr{H}$ is an infinite dimensional vector space over $\mathbb{Q}$, and we denote by $L(\mathscr{H})$ the ring of linear transforms on $\mathscr{H}$.

Lemma 8. *Suppose that $I - \varphi\lambda$ and $I - \psi\lambda$ are left coprime or right coprime, $h = \psi - \varphi$ is of finite rank, and $W = \mathrm{Im}(h)$. Then there exists a finite dimensional space $\widetilde{W}$ containing W such that $(\varphi^k - \psi^k)(\mathscr{H}) \subset \widetilde{W}$ with $k = 1, 2, \ldots$.*

Proof. First we assume that if $I - \varphi\lambda$ and $I - \psi\lambda$ are right coprime, then by Proposition 7 we have $X_0, X_1, \ldots, X_s \in L(\mathscr{H})$ such that

$$hX_m + \varphi hX_{m-1} + \varphi^2 hX_{m-2} + \cdots + \varphi^m hX_0 + \varphi^{m+1} = 0. \tag{21}$$

Write $\widetilde{W} = W + \varphi W + \varphi^2 W + \cdots + \varphi^m W$; $\widetilde{W}$ is an invariant subspace of $\mathscr{H}$ under h. It is not very hard to check that $\varphi(\widetilde{W}) = \varphi^{m+1}W + \varphi W + \varphi^2 W + \cdots + \varphi^m W \subset \widetilde{W}$. So $\widetilde{W}$ is invariant under the operator φ.

We use induction to prove the conclusion.

If $k = 1$, then $(\varphi^k - \psi^k)(\mathscr{H}) = W$.

Suppose that for $k \le l$ we have $(\varphi^k - \psi^k)(\mathscr{H}) \subset W + \varphi W + \cdots + \varphi^m W$. Then for $k = l + 1$, we have $(\varphi^k - \psi^k)(\mathscr{H}) = \varphi(\varphi^l - \psi^l)(\mathscr{H}) - h\psi^l(\mathscr{H}) \subset \varphi(\widetilde{W}) + W \subset \widetilde{W}$. We get the conclusion in this case.

Now we assume that $I - \varphi\lambda$ and $I - \psi\lambda$ are left coprime; then there exist $s \in \mathbb{Z}$ and $x_0, x_1, \ldots, x_s \in L(\mathscr{H})$ such that

$$x_s h + x_{s-1} h\varphi + x_{s-2} h\varphi^2 + \cdots + x_0 h\varphi^s + \varphi^{s+1} = 0, \tag{22}$$

holds. Take

$$\begin{gathered} \widehat{W} = W + x_0 W + \cdots + x_s W, \\ \widetilde{W} = \widehat{W} + \varphi\widehat{W} + \varphi^2\widehat{W} + \cdots + \varphi^s\widehat{W}. \end{gathered} \tag{23}$$

Notice when $l \ge s + 1$, $\varphi^l(\mathscr{H}) \subset x_0 W + x_1 W + \cdots + x_s W$.

We use induction to prove the conclusion.

If $k = 1$, then $(\varphi^k - \psi^k)(\mathscr{H}) = W$.

Suppose that for $k \le l$ we have $(\varphi^k - \psi^k)(\mathscr{H}) \subset W + \varphi W + \cdots + \varphi^m W$. Then for $k = l + 1$, we have $(\varphi^k - \psi^k)(\mathscr{H}) = \varphi^l h(\mathscr{H}) - (\varphi^l - \psi^l)\psi(\mathscr{H}) \subset \varphi^l(W) + \widetilde{W} \subset \widetilde{W}$.

The proof is complete. □

Now we will give the proof of Theorem 5.

Proof. If $I - \varphi\lambda$ and $I - \psi\lambda$ are coprime, by Lemma 8, there is a finite dimensional space $\widetilde{W}$ such that

$$\left(\psi^k - \varphi^k\right)(\mathscr{H}) \subset \widetilde{W}, \tag{24}$$

and we have $\varphi(\widetilde{W}) \subset \widetilde{W}$. Define

$$\widetilde{\varphi} = \mathrm{Pr}(\varphi)|_{\widetilde{W}} : \widetilde{W} \longrightarrow \widetilde{W}. \tag{25}$$

Suppose $q = \mathrm{rank}(\varphi - \psi)$; we denote by I_q the identity operator. Then $p(z) = \det(I_q - z\widetilde{\varphi}) \in \mathbb{Q}[z]$. For any $i, j \in \{1, 2, \ldots, q\}$, we have $\sum_{n\ge 0} \alpha_i \varphi^n(u_j) z^n = f_{ij}(z)/p(z)$ for some $f_{ij}(z) \in \mathbb{Q}[z]$. For more details see [5]. So by Lemma 4, $\det(\mathbf{I} - z\mathbf{M}_\varphi(\overline{\alpha}; \overline{u}))$ is rational; we get the conclusion. □

Example 9. Suppose $\mathscr{H}$ is the ring of countable infinite matrix with finite nonzero entries in each column.

Let

$$\varphi = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ A & 0 & 0 & \cdots \\ 0 & A & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \qquad h = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$
$$\psi = \varphi + h, \tag{26}$$

where $A = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and $\alpha_1, \alpha_2, \alpha_3, \ldots$ are three-dimensional row vectors. We see that $(\varphi - h)^3 = 0$. So $I - \psi\lambda$ and $I - \varphi\lambda$ are left coprime. It is easy to check that

$$\Delta_{(\varphi,\psi)} = \frac{1 - 2z}{1 - z}. \tag{27}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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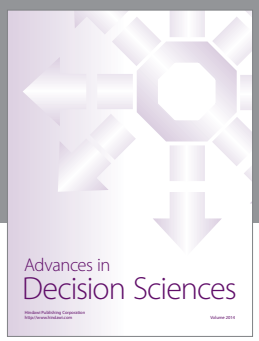

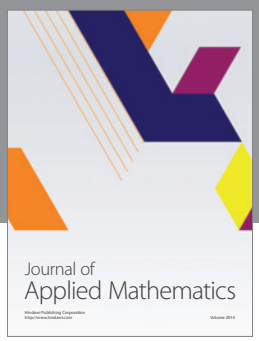

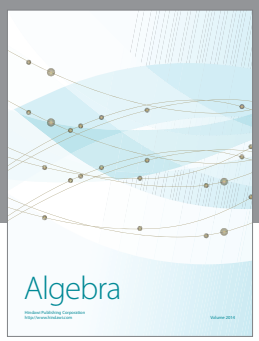

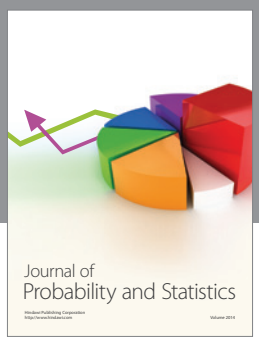

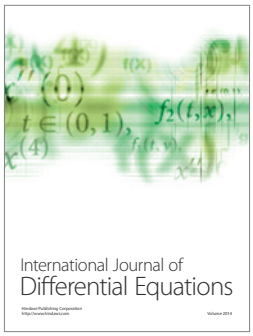

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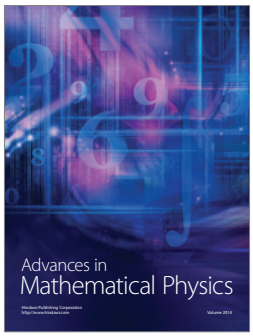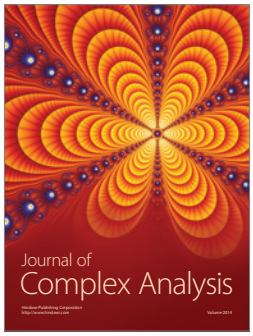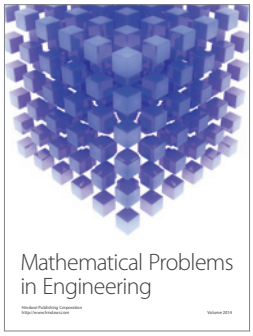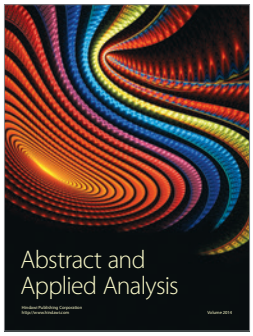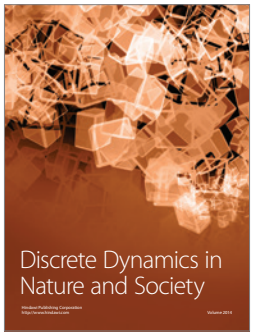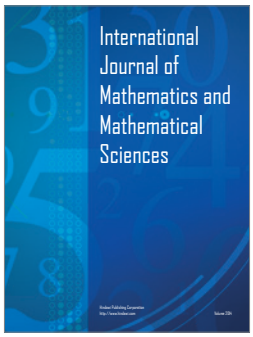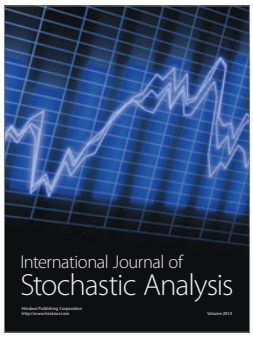
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