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Research Article

H_∞ Cluster Synchronization for a Class of Neutral Complex Dynamical Networks with Markovian Switching

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H_∞ cluster synchronization problem for a class of neutral complex dynamical networks (NCDNs) with Markovian switching is investigated in this paper. Both the retarded and neutral delays are considered to be interval mode dependent and time varying. The concept of H_∞ cluster synchronization is proposed to quantify the attenuation level of synchronization error dynamics against the exogenous disturbance of the NCDNs. Based on a novel Lyapunov functional, by employing some integral inequalities and the nature of convex combination, mode delay-range-dependent H_∞ cluster synchronization criteria are derived in the form of linear matrix inequalities which depend not only on the disturbance attenuation but also on the initial values of the NCDNs. Finally, numerical examples are given to demonstrate the feasibility and effectiveness of the proposed theoretical results.

1. Introduction

During the past decades, the research on the complex dynamical networks (CDNs) has attracted extensive attention of scientific and engineering researchers in all fields domestic and overseas since the pioneering work of Watts and Strogatz [1]. One of the reasons is that the complex networks have extensively existed in many practical applications, such as ecosystems, the Internet, scientific citation web, biological neural networks, and large scale robotic system; see, for example, [2–4]. It should be noted that the synchronization phenomena of CDNs have been paid more attention to and intensively have been investigated in various different fields; please refer to [5–10] and references therein for more details.

Since time delay inevitably exists and has become an important issue in studying the CDNs, synchronization problems for complex networks with time delays have gained increasing research attention and considerable progress has been made; see, for example, [5–16] and references therein for more details. However, in some practical applications, past change rate of the state variables affects the dynamics of nodes in the networks. This kind of complex dynamical network is termed as neutral complex dynamical network (NCDN), which contains delays both in its states and in the derivatives of its states. There are some results about the synchronization design problem for neutral systems [17–21]. In these works, [18, 19] had studied the synchronization control for a kind of master-response setup and further extended to the case of neutral-type neural networks with stochastic perturbation. References [17, 20] had researched the synchronization problem for a class of complex networks with neutral-type coupling delays. Reference [21] had investigated the robust global exponential synchronization problem for an array of neutral-type neural networks. However, much fewer results have been proposed for neutral complex dynamical networks (NCDNs) compared with the rich results for CDNs with only discrete delays.

Recently, as a special synchronization on CDNs, cluster synchronization has been observed in biological science, distributed computation, and social contact networks. Because most of these networks have the clustering characteristic, many individuals maintain close contact with others in a same cluster, while only a few individuals link with an outside cluster. Hence, the individuals are synchronized inside the same cluster, but there is no synchronization among the clusters. Many researchers have made a lot of progress on the cluster synchronization problem; see, for example, [22–26]. In [24], cluster synchronization criteria are proposed for

the coupled Josephson equation by constructing different coupling schemes. Then, in [26], a coupling scheme with cooperative and competitive weigh couplings is used to realize cluster synchronization for connected chaotic networks. In [22], cluster synchronization in an array of hybrid coupled neural networks with delays has been investigated and a new method is proposed to realize cluster synchronization by constructing a special coupling matrix. Besides, in the latest two years, cluster synchronization is considered for an array of coupled stochastic delayed neural networks by using the pinning control strategy in [23]. Linear pinning control schemes are given for cluster mixed synchronization of complex networks with community structure and nonidentical nodes in [25]. However, most of the research results in general complex networks ensure global or asymptotical synchronization, but the external disturbance is always existent, which may cause complex networks to diverge or oscillate. Therefore it is imperative to enhance the anti-interference ability of the system. To our knowledge, not much has been done for H_∞ cluster synchronization for continuous-time complex dynamical networks with neutral time delays and Markovian switching. The purpose of this paper is to minimize this gap. In addition, due to the complexity of high-order and large-scale networks, network mode switching is also a universal phenomenon in CDNs of the actual systems, and sometimes the network has finite modes that switch from one to another with certain transition rate; then such switching can be governed by a Markovian chain. The stability and synchronization problem of complex networks and neural networks with Markovian jump parameters and delays are investigated in [15, 27–30] and references therein. Motivated by the above analysis, the H_∞ cluster synchronization problem for a class of NCDNs with Markovian switching and mode-dependent time-varying delays is investigated in this paper. The addressed NCDNs consist of M modes and the networks switch from one mode to another according to a Markovian chain.

In this paper, H_∞ cluster synchronization of the NCDNs with Markovian jump parameters is studied for the first time, which is first introduced to quantify the attenuation level of synchronization error dynamics against the exogenous disturbance of NCDNs with Markovian switching. It is assumed that the neutral and retarded delays are interval mode dependent and time varying. By utilizing a new augmented Lyapunov functional, H_∞ cluster synchronization criteria, which depend on interval mode-dependent delays, disturbance attenuation lever, and the initial values of NCDNs, are derived based on the Lyapunov stability theory, integral matrix inequalities, and convex combination. All the proposed results are in terms of LMIs that can be solved numerically, which are proved to be effective in numerical examples.

The remainder of the paper is organized as follows. Section 2 presents the problem and preliminaries. Section 3 gives the main results, which are then verified by numerical examples in Section 4. The paper is concluded in Section 5.

Notations. The following notations are used throughout the paper. $\mathbb{R}^n$ denotes the n dimensional Euclidean space and $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ matrices. $X < Y$ ($X > Y$), where X and Y are both symmetric matrices, meaning that $X - Y$ is negative (positive) definite. I is the identity matrix with proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathscr{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector v, and $\|v\| = (v^T v)^{1/2}$, while $\|A\|$ is spectral norm of matrix A and $\|A\| = [\lambda_{\max}(A^T A)]^{1/2}$. $\lambda_{\max(\min)}(A)$ is the eigenvalue of matrix A with maximum (minimum) real part. The Kronecker product of matrices $P \in \mathbb{R}^{m\times n}$ and $Q \in \mathbb{R}^{p\times q}$ is a matrix in $\mathbb{R}^{mp\times nq}$ which is denoted by $P \otimes Q$. Let $\varsigma > 0$ and $C([-\varsigma, 0], \mathbb{R}^n)$ denotes the family of continuous function φ, from $[-\varsigma, 0]$ to $\mathbb{R}^n$ with the norm $|\varphi| = \sup_{-\varsigma\le\theta\le 0}\|\varphi(\theta)\|$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Problem Statement and Preliminaries

Given a complete probability space $\{\Omega, \mathscr{F}, \mathbf{P}\}$ where Ω is the sample space, $\mathscr{F}$ is the algebra of events and $\mathbf{P}$ is the probability measure defined on $\mathscr{F}$. Let $\{r(t), t \ge 0\}$ be a homogeneous and right-continuous Markov chain taking values in a finite state space $S = \{1, 2, 3, \ldots, M\}$ with a generator $\Upsilon = (\gamma_{ij})_{M\times M}$, $i, j \in S$, which is given by

$$P\left(r(t+\Delta t) = j \mid r(t) = i\right) = \begin{cases} \gamma_{ij}\Delta t + o(\Delta t) & i \ne j \\ 1 + \gamma_{ii}\Delta t + o(\Delta t) & i = j, \end{cases} \tag{1}$$

where $\Delta t > 0$, $\lim_{\Delta t\to 0}(o(\Delta t)/\Delta t) = 0$, $\gamma_{ij} \ge 0$ $(i, j \in S, i \ne j)$ is the transition rate from mode i to j and, for any state or mode $i \in S$, it satisfies

$$\gamma_{ii} = -\sum_{j=1, j\ne i}^{M} \gamma_{ij}, \qquad \eta = \max_{i\in S}\{-\gamma_{ii}\}. \tag{2}$$

Moreover, it is assumed that $r(t)$ is irreducible and available at time t.

The following neutral complex dynamical network (NCDN) consisting of N identical nodes with Markovian jump parameters and interval time-varying delays over the space $\{\Omega, \mathscr{F}, \mathbf{P}\}$ is investigated in this paper:

$$\begin{aligned}
&\dot{x}_k(t) - C(r(t))\,\dot{x}_k(t-\tau(t, r(t))) \\
&\quad = A(r(t))\,x_k(t) + B(r(t))\,x_k(t-d(t, r(t))) \\
&\qquad + \sum_{l=1}^{N} g_{kl}^{(1)}(r(t))\,\Gamma_1(r(t))\,x_l(t) \\
&\qquad + \sum_{l=1}^{N} g_{kl}^{(2)}(r(t))\,\Gamma_2(r(t))\,x_l(t-d(t, r(t))) \\
&\qquad + \sum_{l=1}^{N} g_{kl}^{(3)}(r(t))\,\Gamma_3(r(t))\,\dot{x}_l(t-\tau(t, r(t))) \\
&\qquad + D(r(t))\,f_1(x_k(t)) \\
&\qquad + E(r(t))\,f_2(x_k(t-d(t, r(t)))) \\
&\qquad + F(r(t))\,f_3(\dot{x}_k(t-\tau(t, r(t)))) \\
&\qquad + H_k(r(t))\,\omega_k(t),
\end{aligned} \tag{3}$$

$$z_k(t) = L(r(t))\, x_k(t), \tag{4}$$

where $x_k(t) = (x_{k1}(t), x_{k2}(t), \ldots, x_{kn}(t))^T \in \mathbb{R}^n$ and $z_k(t) = (z_{k1}(t), z_{k2}(t), \ldots, z_{kn}(t))^T \in \mathbb{R}^n$ are state variable and the controlled output of the node $k \in \{1, 2, \ldots, N\}$, respectively. $\omega_k(t) \in \mathbb{R}$ is the exogenous disturbance input. $r(t)$ describes the evolution of the mode. $A(r(t))$, $B(r(t))$, $C(r(t))$, $D(r(t))$, $E(r(t))$, and $F(r(t)) \in \mathbb{R}^{n\times n}$ represent the connection weight matrices and the delayed connection weight matrices with real values in mode $r(t)$. $H_k(r(t)) \in \mathbb{R}^n$ $(k = 1, 2, \ldots, N)$ is the disturbance matrix in mode $r(t)$. $L(r(t)) \in \mathbb{R}^{n\times n}$ is a parametric matrix in mode $r(t)$. In this paper, these parametric matrices of NCDN (3) and (4) are known constant matrices in certain mode $r(t)$. $f_1, f_2, f_3 : \mathbb{R}^n \to \mathbb{R}^n$ are continuously nonlinear vector functions which are with respect to the current state $x(t)$, the delayed state $x(t - d(t, r(t)))$, and the neutral delay state $\dot{x}_k(t-\tau(t, r(t)))$. $\Gamma_1(r(t)) \in \mathbb{R}^{n\times n}$, $\Gamma_2(r(t)) \in \mathbb{R}^{n\times n}$, and $\Gamma_3(r(t)) \in \mathbb{R}^{n\times n}$ represent the inner-coupling matrices linking between the subsystems in mode $r(t)$. $G^{(1)}(r(t)) = [g_{kl}^{(1)}]_{N\times N}$, $G^{(2)}(r(t)) = [g_{kl}^{(2)}]_{N\times N}$, and $G^{(3)}(r(t)) = [g_{kl}^{(3)}]_{N\times N}$ are the coupling configuration matrices of the networks representing the coupling strength and the topological structure of the NCDNs in mode $r(t)$, in which $g_{kl}^{(m)}$ is defined as follows. If there exists a connection between kth and lth $(k \neq l)$ nodes, then $g_{kl}^{(m)}(r(t)) = g_{lk}^{(m)}(r(t)) > 0$; otherwise $g_{kl}^{(m)}(r(t)) = g_{lk}^{(m)}(r(t)) = 0$ and

$$\begin{aligned} g_{kk}^{(m)}(r(t)) &= -\sum_{l=1, l\neq k}^{N} g_{kl}^{(m)}(r(t)) = -\sum_{l=1, l\neq k}^{N} g_{lk}^{(m)}(r(t)), \\ &\qquad m = 1, 2, 3;\ k = 1, 2, \ldots, N. \end{aligned} \tag{5}$$

$\tau(t, r(t))$ and $d(t, r(t))$ denote the mode-dependent time-varying neutral delay and retarded delay, respectively. They are assumed to satisfy

$$\begin{gathered} 0 \le \tau_{1i} \le \tau_i(t) \le \tau_{2i} \le \overline{\tau} = \max_{i\in S}\{\tau_{2i}\}, \\ \dot{\tau}_i(t) \le \nu_i < 1, \\ 0 \le d_{1i} \le d_i(t) \le d_{2i}, \quad \text{when } r(t) = i \in S, \end{gathered} \tag{6}$$

where τ_{1i}, τ_{2i}, d_{1i}, and d_{2i} are real constant scalars and $\varsigma = \max_{i\in S}\{\tau_{2i}, d_{2i}\}$.

The nonlinear vector functions, f_1, f_2, and f_3, are assumed to satisfy the following sector-bounded condition [31]:

$$\begin{aligned} &\left[f_1(x) - f_1(y) - W_1^{(1)}(x-y)\right]^T \\ &\quad \times \left[f_1(x) - f_1(y) - W_2^{(1)}(x-y)\right] \le 0, \quad \forall x, y \in \mathbb{R}^n, \\ &\left[f_2(x) - f_2(y) - W_1^{(2)}(x-y)\right]^T \\ &\quad \times \left[f_2(x) - f_2(y) - W_2^{(2)}(x-y)\right] \le 0, \quad \forall x, y \in \mathbb{R}^n, \\ &\left[f_3(x) - f_3(y) - W_1^{(3)}(x-y)\right]^T \\ &\quad \times \left[f_3(x) - f_3(y) - W_2^{(3)}(x-y)\right] \le 0, \quad \forall x, y \in \mathbb{R}^n, \end{aligned} \tag{7}$$

where $W_1^{(l)}$ and $W_2^{(l)}$, $l = 1, 2, 3$, are two constant matrices with $W_2^{(l)} - W_1^{(l)} \ge 0$. Such a description of nonlinear functions has been exploited in [32–34] and is more general than the commonly used Lipschitz conditions, which would be possible to reduce the conservatism of the main results caused by quantifying the nonlinear functions via a matrix inequality technique.

For simplicity of notations, we denote $A(r(t))$, $B(r(t))$, $C(r(t))$, $D(r(t))$, $E(r(t))$, $F(r(t))$, $G^{(m)}(r(t))$, $\Gamma_m(r(t))$, $(m = 1, 2, 3)$, $H_k(r(t))$, and $L(r(t))$ by A_i, B_i, C_i, D_i, E_i, F_i, $G_i^{(m)}$, Γ_{mi}, $(m = 1, 2, 3)$, H_{ki}, and L_i for $r(t) = i \in S$. By utilizing the Kronecker product of the matrices, (3) and (4) can be written in a more compact form as

$$\begin{aligned} \dot{x}(t) &= \mathbb{A}_i x(t) + \mathbb{B}_i x(t - d_i(t)) \\ &\quad + \mathbb{C}_i \dot{x}(t - \tau_i(t)) + \mathbb{D}_i F_1(x(t)) \\ &\quad + \mathbb{E}_i F_2(x(t - d_i(t))) \\ &\quad + \mathbb{F}_i F_3(\dot{x}(t - \tau_i(t))) + \mathbb{H}_i \omega(t), \end{aligned} \tag{8}$$

$$z(t) = \mathbb{L}_i x(t), \tag{9}$$

where

$$\begin{aligned} \mathbb{A}_i &= I_N \otimes A_i + G_i^{(1)} \otimes \Gamma_{1i}, \\ \mathbb{B}_i &= I_N \otimes B_i + G_i^{(2)} \otimes \Gamma_{2i}, \\ \mathbb{C}_i &= I_N \otimes C_i + G_i^{(3)} \otimes \Gamma_{3i}, \\ \mathbb{D}_i &= I_N \otimes D_i, \qquad \mathbb{E}_i = I_N \otimes E_i, \\ \mathbb{F}_i &= I_N \otimes F_i, \qquad \mathbb{L}_i = I_N \otimes L_i, \\ \mathbb{H}_i &= \operatorname{diag}\{H_{1i}, H_{2i}, \ldots, H_{Ni}\}, \\ x(t) &= \operatorname{col}\{x_1(t), x_2(t), \ldots, x_N(t)\}, \\ x(t - d_i(t)) &= \operatorname{col}\{x_1(t - d_i(t)), x_2(t - d_i(t)), \ldots, \\ &\qquad x_N(t - d_i(t))\}, \\ \dot{x}(t - \tau_i(t)) &= \operatorname{col}\{\dot{x}_1(t - \tau_i(t)), \dot{x}_2(t - \tau_i(t)), \ldots, \\ &\qquad \dot{x}_N(t - \tau_i(t))\}, \\ F_1(x(t)) &= \operatorname{col}\{f_1(x_1(t)), f_1(x_2(t)), \ldots, f_1(x_N(t))\}, \\ F_2(x(t - d_i(t))) &= \operatorname{col}\{f_2(x_1(t - d_i(t))), f_2(x_2(t - d_i(t))), \ldots, \\ &\qquad f_2(x_N(t - d_i(t)))\}, \\ F_3(\dot{x}(t - \tau_i(t))) &= \operatorname{col}\{f_3(\dot{x}_1(t - \tau_i(t))), f_3(\dot{x}_2(t - \tau_i(t))), \ldots, \\ &\qquad f_3(\dot{x}_N(t - \tau_i(t)))\}, \\ \omega(t) &= \operatorname{col}\{\omega_1(t), \omega_2(t), \ldots, \omega_N(t)\}, \\ z(t) &= \operatorname{col}\{z_1(t), z_2(t), \ldots, z_N(t)\}. \end{aligned} \tag{10}$$

Assumption 1 (see [22]). The coupling matrix $G_i^{(m)}$ can be expressed in the following form:

$$G_i^{(m)} = \begin{bmatrix} N_{i11}^{(m)} & N_{i12}^{(m)} & \cdots & N_{i1k}^{(m)} \\ N_{i21}^{(m)} & N_{i22}^{(m)} & \cdots & N_{i2k}^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ N_{ik1}^{(m)} & N_{ik2}^{(m)} & \cdots & N_{ikk}^{(m)} \end{bmatrix}, \quad m = 1, 2, 3. \tag{11}$$

It should be especially emphasized that we do not assume that the coupling matrix is symmetric or diagonal. However, most of the former works about network synchronization are based on symmetric or diagonal coupling matrix.

Before moving onto the main results, some definitions and lemmas are introduced below.

Definition 2 (see [35]). Define operator $\mathfrak{D} : C([-\varsigma, 0], \mathbb{R}^n) \to \mathbb{R}^n$ by $\mathfrak{D}(x_t) = x(t) - Cx(t-\tau)$. $\mathfrak{D}$ is said to be stable if the homogeneous difference equation

$$\begin{gathered} \mathfrak{D}(x_t) = 0, \quad t \geq 0, \\ x_0 = \psi \in \{\phi \in C([-\varsigma, 0], \mathbb{R}^n) : \mathfrak{D}\phi = 0\} \end{gathered} \tag{12}$$

is uniformly asymptotically stable. In this paper, that is, $\|I_N \otimes C_i + G_i^{(3)} \otimes C_i\| < 1$.

Definition 3 (see [36]). Define the stochastic Lyapunov-Krasovskii function of the NCDNs (3) and (4) as $V(x(t), r(t) = i, t > 0) = V(x(t), i, t)$ where its infinitesimal generator is defined as

$$\begin{aligned} &\Gamma V(x(t), i, t) \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} [\mathscr{E}\{V(x(t+\Delta t), r(t+\Delta t), \\ &\qquad t+\Delta t) \mid x(t) = x, r(t) = i\} \\ &\qquad - V(x(t), i, t)] \\ &= \frac{\partial}{\partial t} V(x(t), i, t) + \frac{\partial}{\partial x} V(x(t), i, t)\, \dot{x}(t) \\ &\quad + \sum_{j=1}^{N} \pi_{ij} V(x(t), j, t). \end{aligned} \tag{13}$$

Definition 4 (see [26]). A network with N nodes realizes cluster synchronization if the N nodes are split into several clusters, such as $\{(1, 2, \ldots, m_1), (m_1 + 1, m_1 + 2, \ldots, m_2), \ldots, (m_{k-1} + 1, m_{k-1} + 2, \ldots, m_k), m_0 = 0, m_k = N, m_{j-1} < m_j\}$, and the nodes in the same cluster synchronize with one another (i.e., for the states $x_i(t)$ and $x_j(t)$ of arbitrary nodes i and j in the same cluster, $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$ holds). The set

$$\begin{aligned} \mathcal{S} = \{&x = (x_1(s), x_2(s), \ldots, x_N(s)) : x_1(s) = x_2(s) = \cdots \\ &= x_{m_1}(s), x_{m_1+1}(s) = x_{m_1+2}(s) = \cdots \\ &= x_{m_2}(s), \ldots, x_{m_{k-1}+1}(s) = x_{m_{k-1}+2}(s) = \cdots = x_{m_k}(s)\} \end{aligned} \tag{14}$$

is called the cluster synchronization manifold.

Lemma 5 (see [37]). *Let G be an $N \times N$ matrix in the set $T(\mathfrak{R}, K)$, where $\mathfrak{R}$ denotes a ring and $T(\mathfrak{R}, K) = \{$the set of matrices with entries $\mathfrak{R}$ such that the sum of the entries in each row is equal to K for some $K \in \mathfrak{R}\}$. Then the $(N-1) \times (N-1)$ matrix X satisfies $MG = XM$, where $X = MGJ$,*

$$\begin{gathered} M = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{(N-1)\times N}, \\ J = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ & & \ddots & & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{N\times(N-1)}. \end{gathered} \tag{15}$$

Furthermore, the matrix X can be rewritten explicitly as follows:

$$X_{p,q} = \sum_{k=1}^{q} \left(G_{p,k} - G_{p+1,k}\right), \quad \text{for } p, q \in \{1, 2, \ldots, N-1\}. \tag{16}$$

Lemma 6. *Under Assumption 1, the $(N-k) \times (N-k)$ matrix $X_i^{(m)}$ satisfies $\widetilde{M} G_i^{(m)} = X_i^{(m)} \widetilde{M}$, $m = 1, 2, 3$, where*

$$\begin{gathered} N_i^{(m)} = \begin{bmatrix} N_{i11}^{(m)} & & & \\ & N_{i22}^{(m)} & & \\ & & \ddots & \\ & & & N_{ikk}^{(m)} \end{bmatrix}_{N\times N}, \\ \widetilde{M} = \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_k \end{bmatrix}_{(N-k)\times N}, \\ \widetilde{J} = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}_{N\times(N-k)}. \end{gathered} \tag{17}$$

And $X_i^{(m)} = \widetilde{M} N_i^{(m)} \widetilde{J}$, $N_{pp}^{(m)} \in \mathbb{R}^{m_p \times m_p}$, $M_p \in \mathbb{R}^{(m_p-1)\times m_p}$, $J_p \in \mathbb{R}^{m_p \times (m_p-1)}$, and $p = 1, 2, \ldots, k$.

Proof. From Assumption 1 and Lemma 5, it can be easily obtained that

$$\begin{aligned} &\widetilde{M} G_i^{(m)} \\ &= \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_k \end{bmatrix} \\ &\quad \times \begin{bmatrix} N_{i11}^{(m)} & N_{i12}^{(m)} & \cdots & N_{i1k}^{(m)} \\ N_{i21}^{(m)} & N_{i22}^{(m)} & \cdots & N_{i2k}^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ N_{ik1}^{(m)} & N_{ik2}^{(m)} & \cdots & N_{ikk}^{(m)} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} M_1 N_{i11}^{(m)} & & & \\ & M_2 N_{i22}^{(m)} & & \\ & & \ddots & \\ & & & M_k N_{ikk}^{(m)} \end{bmatrix}$$
$$= \begin{bmatrix} M_1 N_{i11}^{(m)} J_1 M_1 & & & \\ & M_2 N_{i22}^{(m)} J_2 M_2 & & \\ & & \ddots & \\ & & & M_k N_{ikk}^{(m)} J_k M_k \end{bmatrix}$$
$$= \widetilde{M} N_i^{(m)} \widetilde{J} \widetilde{M} = X_i^{(m)} \widetilde{M}. \tag{18}$$

This completes the proof. □

Lemma 7 (see [22]). *$x \in \mathcal{S}$ if and only if $\mathcal{E}\{\|\mathbf{M}x(t)\|^2\} = 0$, $t \to \infty$, where $\mathbf{M} = \widetilde{M} \otimes I_N$.*

Proof. Consider

$$\mathcal{E}\left\{\|\mathbf{M}x(t)\|^2\right\} = \mathcal{E}\left\{\sum_{l=1}^{m_1-1} \|x_l(t) - x_{l+1}(t)\|^2 + \sum_{l=m_1+1}^{m_2-1} \|x_l(t) - x_{l+1}(t)\|^2 + \cdots + \sum_{l=m_{k-1}+1}^{m_k-1} \|x_l(t) - x_{l+1}(t)\|^2\right\}. \tag{19}$$

By Definition 4, it completes the proof. □

Definition 8. The neutral complex dynamical networks (3) and (4) are H_∞ cluster synchronization with a disturbance attenuation δ and symmetric positive matrix $Y > 0$, if the following condition is satisfied:

$$\int_0^\infty \|\mathbf{M}z(t)\|^2 dt \le \delta^2 \left\{\int_0^\infty \|\omega(t)\|^2 dt + x^T(0) Y x(0)\right\}. \tag{20}$$

The index δ is called disturbance attenuation and used to quantify the attenuation level of synchronization error dynamics against exogenous disturbances. It is noticed that (20) depends not only on the attenuation level but also on the initial values of complex networks.

Lemma 9 (see [38]). *Given matrices A, B, C, and D with appropriate dimensions and scalar α, by the definition of the Kronecker product, the following properties hold:*

$$\begin{gathered} (\alpha A) \otimes B = A \otimes (\alpha B), \\ (A + B) \otimes C = A \otimes C + B \otimes C, \\ (A \otimes B)(C \otimes D) = (AC) \otimes (BD), \\ (A \otimes B)^T = A^T \otimes B^T. \end{gathered} \tag{21}$$

Lemma 10 (see [39, 40]). *For any constant matrix $H = H^T > 0$ and scalars $\tau_2 > \tau_1 > 0$ such that the following integrations are well defined, then*

(a)
$$-(\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} x^T(s) H x(s)\, ds \le -\left[\int_{t-\tau_2}^{t-\tau_1} x^T(s)\, ds\right] H \left[\int_{t-\tau_2}^{t-\tau_1} x(s)\, ds\right], \tag{22}$$

(b)
$$-\frac{1}{2}\left(\tau_2^2 - \tau_1^2\right) \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^{t} x^T(s) H x(s)\, ds\, d\theta \le -\left[\int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^{t} x^T(s)\, ds\, d\theta\right] \times H \left[\int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^{t} x(s)\, ds\, d\theta\right]. \tag{23}$$

Lemma 11 (see [41]). *Supposing that $0 \le \tau_m \le \tau(t) \le \tau_M$, Ξ_1, Ξ_2, and Ω are constant matrices of appropriate dimensions, then*

$$(\tau(t) - \tau_m)\Xi_1 + (\tau_M - \tau(t))\Xi_2 + \Omega < 0 \tag{24}$$

if and only if $(\tau_M - \tau_m)\Xi_1 + \Omega < 0$ and $(\tau_M - \tau_m)\Xi_2 + \Omega < 0$ hold.

3. Main Results

In this section, sufficient conditions are presented to ensure H_∞ cluster synchronization for the neutral complex dynamical network (NCDN) (3) and (4).

3.1. H_∞ Cluster Synchronization Analysis

Theorem 12. *Given the transition rate matrix Υ, the initial positive definite matrix $Y = Y^T > 0$, constant scalars τ_{1i}, τ_{2i}, ν_i, d_{1i}, d_{2i}, and τ_{mi}, d_{mi} satisfying $\tau_{1i} < \tau_{mi} < \tau_{2i}$, $d_{1i} < d_{mi} < d_{2i}$, respectively, the NCDN systems (3) and (4) with sector-bounded condition (7) are H_∞ cluster synchronization with a disturbance attenuation lever δ if $\|(I_N + G_i^{(3)}) \otimes C_i\| < 1$ and there exist $(N-k)n \times (N-k)n$ symmetric positive matrices $P_i > 0$, $(i \in S)$, $Q_j > 0$, $(j = 1, 2, \ldots, 6)$, $R_k > 0$, $(k = 1, 2, \ldots, 7)$, $T_l > 0$, $U_m > 0$, and $V_n > 0$, $(l, m, n = 1, 2, \ldots, 6)$ for any scalars $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ such that the following linear matrix inequalities hold:*

$$\begin{aligned} &\Omega_{i1} + \frac{1}{2}\Theta_{i0} < 0, && \Omega_{i2} + \frac{1}{2}\Theta_{i0} < 0, \\ &\Omega_{i3} + \frac{1}{2}\Theta_{i0} < 0, && \Omega_{i4} + \frac{1}{2}\Theta_{i0} < 0, \\ &\Pi_{i1} + \frac{1}{2}\Theta_{i0} < 0, && \Pi_{i2} + \frac{1}{2}\Theta_{i0} < 0, \\ &\Pi_{i3} + \frac{1}{2}\Theta_{i0} < 0, && \Pi_{i4} + \frac{1}{2}\Theta_{i0} < 0, \end{aligned} \tag{25}$$

$$V(0) < \delta^2 x^T(0) Y x(0), \tag{26}$$

where

$$\begin{aligned}\Theta_{i0} = {} & \sum_{m=1}^{30} E_m \Phi_m E_m^T + \mathscr{L}(\maltese) + \Lambda^T J \Lambda - (E_1 - E_3) \\ & \times U_1 \left(E_1^T - E_3^T\right) - (E_1 - E_{16}) U_4 \left(E_1^T - E_{16}^T\right) \\ & - (\tau_{1i} E_1 - E_{10}) V_1 \left(\tau_{1i} E_1^T - E_{10}^T\right) \\ & - \left[(\tau_{mi} - \tau_{1i}) E_1 - E_{13}\right] V_2 \left[(\tau_{mi} - \tau_{1i}) E_1^T - E_{13}^T\right] \\ & - \left[(\tau_{2i} - \tau_{mi}) E_1 - E_{14}\right] V_3 \left[(\tau_{2i} - \tau_{mi}) E_1^T - E_{14}^T\right] \\ & - (d_{1i} E_1 - E_{22}) V_4 \left(d_{1i} E_1^T - E_{22}^T\right) \\ & - \left[(d_{mi} - d_{1i}) E_1 - E_{25}\right] V_5 \left[(d_{mi} - d_{1i}) E_1^T - E_{25}^T\right] \\ & - \left[(d_{2i} - d_{mi}) E_1 - E_{26}\right] V_6 \left[(d_{2i} - d_{mi}) E_1^T - E_{26}^T\right],\end{aligned} \tag{27}$$

where E_i $\{i = 1, 2, \ldots, 30\}$ *are block entry matrices; that is,*

$$E_4^T = [0 \ \ 0 \ \ 0 \ \ I \ \ 0]. \tag{28}$$

$\mathscr{L}$ *is a linear operator on real square matrices by*

$$\mathscr{L}(A) = A + A^T, \quad \forall A \in \mathbb{R}^{n \times n},$$

$$\begin{aligned}J = {} & \eta(\overline{\tau} - \tau_{1i}) R_1 + R_2 + R_5 + \tau_{1i}^2 U_1 + d_{1i}^2 U_4 \\ & + (\tau_{mi} - \tau_{1i})^2 U_2 + (\tau_{2i} - \tau_{mi})^2 U_3 \\ & + (d_{mi} - d_{1i})^2 U_5 + (d_{2i} - d_{mi})^2 U_6 + \frac{\tau_{1i}^4}{4} V_1 \\ & + \frac{d_{1i}^4}{4} V_4 + \frac{\left(\tau_{mi}^2 - \tau_{1i}^2\right)^2}{4} V_2 + \frac{\left(\tau_{2i}^2 - \tau_{mi}^2\right)^2}{4} V_3 \\ & + \frac{\left(d_{mi}^2 - d_{1i}^2\right)^2}{4} V_5 + \frac{\left(d_{2i}^2 - d_{mi}^2\right)^2}{4} V_6,\end{aligned}$$

$$\begin{aligned}\Lambda = {} & \left(A_i^{\otimes} + \overrightarrow{X_i^{(1)}}\right) E_1^T + \left(C_i^{\otimes} + \overrightarrow{X_i^{(3)}}\right) E_6^T \\ & + \left(B_i^{\otimes} + \overrightarrow{X_i^{(2)}}\right) E_{15}^T + D_i^{\otimes} E_{27}^T + E_i^{\otimes} E_{28}^T \\ & + F_i^{\otimes} E_{29}^T + \mathbf{M} \mathbb{H}_i E_{30}^T,\end{aligned}$$

$$\begin{aligned}\maltese = {} & E_1 \left(P_i D_i^{\otimes} + \epsilon_1 W_1^{(1)^T} + \epsilon_1 W_2^{(1)^T}\right) E_{27}^T \\ & + E_1 P_i D_i^{\otimes} E_{28}^T + E_1 P_i F_i^{\otimes} E_{29}^T + E_1 P_i \mathbb{H}_i E_{30}^T \\ & + E_{15} \left(\epsilon_2 W_1^{(2)^T} + \epsilon_2 W_2^{(2)^T}\right) E_{28}^T \\ & + E_6 \left(\epsilon_3 W_1^{(3)^T} + \epsilon_3 W_2^{(3)^T}\right) E_{29}^T,\end{aligned}$$

$$\begin{aligned}\Phi_1 = {} & \mathscr{L}\left[P_i \left(A_i^{\otimes} + B_i^{\otimes} + C_i^{\otimes} + \overrightarrow{X_i^{(1)}} + \overrightarrow{X_i^{(2)}} + \overrightarrow{X_i^{(3)}}\right)\right. \\ & \left. - \epsilon_1 W_1^{(1)^T} W_2^{(1)}\right] + \sum_{j \in S} \gamma_{ij} P_j + Q_1 + Q_4 \\ & + \tau_{1i}^2 T_1 + d_{1i}^2 T_4 + (\tau_{mi} - \tau_{1i})^2 T_2 + (\tau_{2i} - \tau_{mi})^2 T_3 \\ & + (d_{mi} - d_{1i})^2 T_5 + (d_{2i} - d_{mi})^2 T_6 + L_i^{\otimes^T} L_i^{\otimes},\end{aligned}$$

$$\Phi_3 = Q_2 - R_1, \qquad \Phi_4 = Q_4 - R_2,$$

$$\Phi_5 = -Q_3, \qquad \Phi_6 = -(1 - \nu_i) R_1 - \epsilon_3 W_1^{(3)^T} W_2^{(3)},$$

$$\Phi_7 = R_1 + R_3 - R_2, \qquad \Phi_8 = R_4 - R_3,$$

$$\Phi_9 = -R_4, \qquad \Phi_{10} = -T_1,$$

$$\Phi_{15} = -\epsilon_2 W_1^{(2)^T} W_2^{(2)}, \qquad \Phi_{16} = Q_5 - Q_4,$$

$$\Phi_{17} = Q_6 - Q_5, \qquad \Phi_{18} = -Q_6,$$

$$\Phi_{19} = R_6 - R_5, \qquad \Phi_{20} = R_7 - R_6,$$

$$\Phi_{21} = -R_7, \qquad \Phi_{27} = -2\epsilon_1 I,$$

$$\Phi_{28} = -2\epsilon_2 I, \qquad \Phi_{29} = -2\epsilon_3 I,$$

$$\Phi_{30} = -\delta^2 I,$$

$$\Phi_m = 0, \quad (m = 2, 11, 12, 13, 14, 23, 24, 25, 26),$$

$$A_i^{\otimes} = I_{N-k} \otimes A_i, \qquad B_i^{\otimes} = I_{N-k} \otimes B_i,$$

$$C_i^{\otimes} = I_{N-k} \otimes C_i, \qquad D_i^{\otimes} = I_{N-k} \otimes D_i,$$

$$E_i^{\otimes} = I_{N-k} \otimes E_i, \qquad F_i^{\otimes} = I_{N-k} \otimes F_i,$$

$$L_i^{\otimes} = I_{N-k} \otimes L_i,$$

$$\overrightarrow{X_i^{(m)}} = X_i^{(m)} \otimes \Gamma_{mi}, \quad (m = 1, 2, 3),$$

$$\begin{aligned}\Omega_{i1} = {} & -E_{11} T_2 E_{11}^T - 2(E_{13} - E_{11}) T_2 \left(E_{13}^T - E_{11}^T\right) \\ & - (E_3 - E_2) U_2 \left(E_3^T - E_2^T\right) - 2(E_2 - E_4) \\ & \times U_2 \left(E_2^T - E_4^T\right) - E_{14} T_3 E_{14}^T \\ & - (E_4 - E_5) U_3 \left(E_4^T - E_5^T\right),\end{aligned}$$

$$\begin{aligned}\Omega_{i2} = &-2E_{11}T_2E_{11}^T - (E_{13}-E_{11})T_2(E_{13}^T-E_{11}^T)\\ &-2(E_3-E_2)U_2(E_3^T-E_2^T)-(E_2-E_4)\\ &\times U_2(E_2^T-E_4^T)-E_{14}T_3E_{14}^T\\ &-(E_4-E_5)U_3(E_4^T-E_5^T),\end{aligned}$$

$$\begin{aligned}\Omega_{i3} = &-2E_{12}T_3E_{12}^T - (E_{14}-E_{12})T_3(E_{14}^T-E_{12}^T)\\ &-(E_4-E_2)U_3(E_4^T-E_2^T)-2(E_2-E_5)\\ &\times U_3(E_2^T-E_5^T)-E_{13}T_2E_{13}^T\\ &-(E_3-E_4)U_2(E_3^T-E_4^T),\end{aligned}$$

$$\begin{aligned}\Omega_{i4} = &-E_{12}T_3E_{12}^T - 2(E_{14}-E_{12})T_3(E_{14}^T-E_{12}^T)\\ &-2(E_4-E_2)U_3(E_4^T-E_2^T)\\ &-(E_2-E_5)U_3(E_2^T-E_5^T)\\ &-E_{13}T_2E_{13}^T-(E_3-E_4)U_2(E_3^T-E_4^T),\end{aligned}$$

$$\begin{aligned}\Pi_{i1} = &-E_{23}T_5E_{23}^T - 2(E_{25}-E_{23})T_5(E_{25}^T-E_{23}^T)\\ &-(E_{16}-E_{15})U_5(E_{16}^T-E_{15}^T)\\ &-2(E_{15}-E_{17})U_5(E_{15}^T-E_{17}^T)\\ &-E_{26}T_6E_{26}^T-(E_{17}-E_{18})U_6(E_{17}^T-E_{18}^T),\end{aligned}$$

$$\begin{aligned}\Pi_{i2} = &-2E_{23}T_5E_{23}^T - (E_{25}-E_{23})T_5(E_{25}^T-E_{23}^T)\\ &-2(E_{16}-E_{15})U_5(E_{16}^T-E_{15}^T)\\ &-(E_{15}-E_{17})U_5(E_{15}^T-E_{17}^T)\\ &-E_{26}T_6E_{26}^T-(E_{17}-E_{18})U_6(E_{17}^T-E_{18}^T),\end{aligned}$$

$$\begin{aligned}\Pi_{i3} = &-2E_{24}T_6E_{24}^T - (E_{26}-E_{24})T_6(E_{26}^T-E_{24}^T)\\ &-(E_{17}-E_{15})U_6(E_{17}^T-E_{15}^T)\\ &-2(E_{15}-E_{18})U_6(E_{15}^T-E_{18}^T)-E_{25}T_5E_{25}^T\\ &-(E_{16}-E_{17})U_5(E_{16}^T-E_{17}^T),\end{aligned}$$

$$\begin{aligned}\Pi_{i4} = &-E_{24}T_6E_{24}^T - 2(E_{26}-E_{24})T_6(E_{26}^T-E_{24}^T)\\ &-2(E_{17}-E_{15})U_6(E_{17}^T-E_{15}^T)\\ &-(E_{15}-E_{18})U_6(E_{15}^T-E_{18}^T)-E_{25}T_5E_{25}^T\\ &-(E_{16}-E_{17})U_5(E_{16}^T-E_{17}^T),\end{aligned}$$

$$V(0) = x^T(0)\mathbf{M}^TP_i\mathbf{M}x(0) + \sum_{k=2}^{6}V_k(0),$$

$$\begin{aligned}V_2(0) = &\int_{-\tau_{1i}}^{0} x^T(s)\mathbf{M}^TQ_1\mathbf{M}x(s)\,ds\\ &+\int_{-\tau_{mi}}^{-\tau_{1i}} x^T(s)\mathbf{M}^TQ_2\mathbf{M}x(s)\,ds\\ &+\int_{-\tau_{2i}}^{-\tau_{mi}} x^T(s)\mathbf{M}^TQ_3\mathbf{M}x(s)\,ds\\ &+\int_{-d_{1i}}^{0} x^T(s)\mathbf{M}^TQ_4\mathbf{M}x(s)\,ds\\ &+\int_{-d_{mi}}^{-d_{1i}} x^T(s)\mathbf{M}^TQ_5\mathbf{M}x(s)\,ds\\ &+\int_{-d_{2i}}^{-d_{mi}} x^T(s)\mathbf{M}^TQ_6\mathbf{M}x(s)\,ds,\end{aligned}$$

$$\begin{aligned}V_3(0) = &\int_{-\tau_i(t)}^{-\tau_{1i}} \dot{x}^T(s)\mathbf{M}^TR_1\mathbf{M}\dot{x}(s)\,ds\\ &+\int_{-\tau_{1i}}^{0} \dot{x}^T(s)\mathbf{M}^TR_2\mathbf{M}\dot{x}(s)\,ds\\ &+\int_{-\tau_{mi}}^{-\tau_{1i}} \dot{x}^T(s)\mathbf{M}^TR_3\mathbf{M}\dot{x}(s)\,ds\\ &+\int_{-\tau_{2i}}^{-\tau_{mi}} \dot{x}^T(s)\mathbf{M}^TR_4\mathbf{M}\dot{x}(s)\,ds\\ &+\int_{-d_{1i}}^{0} \dot{x}^T(s)\mathbf{M}^TR_5\mathbf{M}\dot{x}(s)\,ds\\ &+\int_{-d_{mi}}^{-d_{1i}} \dot{x}^T(s)\mathbf{M}^TR_6\mathbf{M}\dot{x}(s)\,ds\\ &+\int_{-d_{2i}}^{-d_{mi}} \dot{x}^T(s)\mathbf{M}^TR_7\mathbf{M}\dot{x}(s)\,ds,\end{aligned}$$

$$\begin{aligned}&V_4(0)\\ &= \int_{-\tau_{1i}}^{0}\int_{\theta}^{0} \tau_{1i}x^T(s)\mathbf{M}^TT_1\mathbf{M}x(s)\,ds\,d\theta\\ &\quad+\int_{-\tau_{mi}}^{-\tau_{1i}}\int_{\theta}^{0} (\tau_{mi}-\tau_{1i})x^T(s)\mathbf{M}^TT_2\mathbf{M}x(s)\,ds\,d\theta\\ &\quad+\int_{-\tau_{2i}}^{-\tau_{mi}}\int_{\theta}^{0} (\tau_{2i}-\tau_{mi})x^T(s)\mathbf{M}^TT_3\mathbf{M}x(s)\,ds\,d\theta\\ &\quad+\int_{-d_{1i}}^{0}\int_{\theta}^{0} d_{1i}x^T(s)\mathbf{M}^TT_4\mathbf{M}x(s)\,ds\,d\theta\\ &\quad+\int_{-d_{mi}}^{-d_{1i}}\int_{\theta}^{0} (d_{mi}-d_{1i})x^T(s)\mathbf{M}^TT_5\mathbf{M}x(s)\,ds\,d\theta\\ &\quad+\int_{-d_{2i}}^{-d_{mi}}\int_{\theta}^{0} (d_{2i}-d_{mi})x^T(s)\mathbf{M}^TT_6\mathbf{M}x(s)\,ds\,d\theta,\end{aligned}$$

$$
\begin{aligned}
&V_5(0)\\
&= \int_{-\tau_{1i}}^{0}\int_{\theta}^{0} \tau_{1i}\dot{x}^T(s)\mathbf{M}^T U_1 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-\tau_{mi}}^{-\tau_{1i}}\int_{\theta}^{0} (\tau_{mi}-\tau_{1i})\dot{x}^T(s)\mathbf{M}^T U_2 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-\tau_{2i}}^{-\tau_{mi}}\int_{\theta}^{0} (\tau_{2i}-\tau_{mi})\dot{x}^T(s)\mathbf{M}^T U_3 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-d_{1i}}^{0}\int_{\theta}^{0} d_{1i}\dot{x}^T(s)\mathbf{M}^T U_4 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-d_{mi}}^{-d_{1i}}\int_{\theta}^{0} (d_{mi}-d_{1i})\dot{x}^T(s)\mathbf{M}^T U_5 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-d_{2i}}^{-d_{mi}}\int_{\theta}^{0} (d_{2i}-d_{mi})\dot{x}^T(s)\mathbf{M}^T U_6 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-\bar{\tau}}^{-\tau_{1i}}\int_{\theta}^{0} \eta\dot{x}^T(s)\mathbf{M}^T R_1 \mathbf{M}\dot{x}(s)\,ds\,d\theta,
\end{aligned}
$$

$$
\begin{aligned}
&V_6(0)\\
&= \int_{-\tau_{1i}}^{0}\int_{\theta}^{0}\int_{\lambda}^{0} \frac{\tau_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_1 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-\tau_{mi}}^{-\tau_{1i}}\int_{\theta}^{0}\int_{\lambda}^{0} \frac{\tau_{mi}^2-\tau_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_2 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-\tau_{2i}}^{-\tau_{mi}}\int_{\theta}^{0}\int_{\lambda}^{0} \frac{\tau_{2i}^2-\tau_{mi}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_3 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-d_{1i}}^{0}\int_{\theta}^{0}\int_{\lambda}^{0} \frac{d_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_4 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-d_{mi}}^{-d_{1i}}\int_{\theta}^{0}\int_{\lambda}^{0} \frac{d_{mi}^2-d_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_5 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-d_{2i}}^{-d_{mi}}\int_{\theta}^{0}\int_{\lambda}^{0} \frac{d_{2i}^2-d_{mi}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_6 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta.
\end{aligned}
\tag{29}
$$

Proof. Construct the Lyapunov functional candidate as follows:

$$
V(x(t),i,t) = \sum_{k=1}^{6} V_k(x(t),i,t), \tag{30}
$$

where

$$
V_1(x(t),i,t) = x^T(t)\mathbf{M}^T P_i \mathbf{M}x(t),
$$

$$
\begin{aligned}
V_2(x(t),i,t) &= \int_{t-\tau_{1i}}^{t} x^T(s)\mathbf{M}^T Q_1 \mathbf{M}x(s)\,ds\\
&\quad + \int_{t-\tau_{mi}}^{t-\tau_{1i}} x^T(s)\mathbf{M}^T Q_2 \mathbf{M}x(s)\,ds\\
&\quad + \int_{t-\tau_{2i}}^{t-\tau_{mi}} x^T(s)\mathbf{M}^T Q_3 \mathbf{M}x(s)\,ds\\
&\quad + \int_{t-d_{1i}}^{t} x^T(s)\mathbf{M}^T Q_4 \mathbf{M}x(s)\,ds\\
&\quad + \int_{t-d_{mi}}^{t-d_{1i}} x^T(s)\mathbf{M}^T Q_5 \mathbf{M}x(s)\,ds\\
&\quad + \int_{t-d_{2i}}^{t-d_{mi}} x^T(s)\mathbf{M}^T Q_6 \mathbf{M}x(s)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
V_3(x(t),i,t) &= \int_{t-\tau_i(t)}^{t-\tau_{1i}} \dot{x}^T(s)\mathbf{M}^T R_1 \mathbf{M}\dot{x}(s)\,ds\\
&\quad + \int_{t-\tau_{1i}}^{t} \dot{x}^T(s)\mathbf{M}^T R_2 \mathbf{M}\dot{x}(s)\,ds\\
&\quad + \int_{t-\tau_{mi}}^{t-\tau_{1i}} \dot{x}^T(s)\mathbf{M}^T R_3 \mathbf{M}\dot{x}(s)\,ds\\
&\quad + \int_{t-\tau_{2i}}^{t-\tau_{mi}} \dot{x}^T(s)\mathbf{M}^T R_4 \mathbf{M}\dot{x}(s)\,ds\\
&\quad + \int_{t-d_{1i}}^{t} \dot{x}^T(s)\mathbf{M}^T R_5 \mathbf{M}\dot{x}(s)\,ds\\
&\quad + \int_{t-d_{mi}}^{t-d_{1i}} \dot{x}^T(s)\mathbf{M}^T R_6 \mathbf{M}\dot{x}(s)\,ds\\
&\quad + \int_{t-d_{2i}}^{t-d_{mi}} \dot{x}^T(s)\mathbf{M}^T R_7 \mathbf{M}\dot{x}(s)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
&V_4(x(t),i,t)\\
&= \int_{-\tau_{1i}}^{0}\int_{t+\theta}^{t} \tau_{1i}x^T(s)\mathbf{M}^T T_1 \mathbf{M}x(s)\,ds\,d\theta\\
&\quad + \int_{-\tau_{mi}}^{-\tau_{1i}}\int_{t+\theta}^{t} (\tau_{mi}-\tau_{1i})x^T(s)\mathbf{M}^T T_2 \mathbf{M}x(s)\,ds\,d\theta\\
&\quad + \int_{-\tau_{2i}}^{-\tau_{mi}}\int_{t+\theta}^{t} (\tau_{2i}-\tau_{mi})x^T(s)\mathbf{M}^T T_3 \mathbf{M}x(s)\,ds\,d\theta\\
&\quad + \int_{-d_{1i}}^{0}\int_{t+\theta}^{t} d_{1i}x^T(s)\mathbf{M}^T T_4 \mathbf{M}x(s)\,ds\,d\theta\\
&\quad + \int_{-d_{mi}}^{-d_{1i}}\int_{t+\theta}^{t} (d_{mi}-d_{1i})x^T(s)\mathbf{M}^T T_5 \mathbf{M}x(s)\,ds\,d\theta\\
&\quad + \int_{-d_{2i}}^{-d_{mi}}\int_{t+\theta}^{t} (d_{2i}-d_{mi})x^T(s)\mathbf{M}^T T_6 \mathbf{M}x(s)\,ds\,d\theta,
\end{aligned}
$$

$$
\begin{aligned}
&V_5(x(t),i,t)\\
&= \int_{-\tau_{1i}}^{0}\int_{t+\theta}^{t} \tau_{1i}\dot{x}^T(s)\mathbf{M}^T U_1 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-\tau_{mi}}^{-\tau_{1i}}\int_{t+\theta}^{t} (\tau_{mi}-\tau_{1i})\dot{x}^T(s)\mathbf{M}^T U_2 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-\tau_{2i}}^{-\tau_{mi}}\int_{t+\theta}^{t} (\tau_{2i}-\tau_{mi})\dot{x}^T(s)\mathbf{M}^T U_3 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-d_{1i}}^{0}\int_{t+\theta}^{t} d_{1i}\dot{x}^T(s)\mathbf{M}^T U_4 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-d_{mi}}^{-d_{1i}}\int_{t+\theta}^{t} (d_{mi}-d_{1i})\dot{x}^T(s)\mathbf{M}^T U_5 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-d_{2i}}^{-d_{mi}}\int_{t+\theta}^{t} (d_{2i}-d_{mi})\dot{x}^T(s)\mathbf{M}^T U_6 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-\bar{\tau}}^{-\tau_{1i}}\int_{t+\theta}^{t} \eta\dot{x}^T(s)\mathbf{M}^T R_1 \mathbf{M}\dot{x}(s)\,ds\,d\theta,\\
&V_6(x(t),i,t)\\
&= \int_{-\tau_{1i}}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \frac{\tau_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_1 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-\tau_{mi}}^{-\tau_{1i}}\int_{\theta}^{0}\int_{t+\lambda}^{t} \frac{\tau_{mi}^2-\tau_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_2 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-\tau_{2i}}^{-\tau_{mi}}\int_{\theta}^{0}\int_{t+\lambda}^{t} \frac{\tau_{2i}^2-\tau_{mi}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_3 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-d_{1i}}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \frac{d_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_4 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-d_{mi}}^{-d_{1i}}\int_{\theta}^{0}\int_{t+\lambda}^{t} \frac{d_{mi}^2-d_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_5 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-d_{2i}}^{-d_{mi}}\int_{\theta}^{0}\int_{t+\lambda}^{t} \frac{d_{2i}^2-d_{mi}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_6 \mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta.
\end{aligned}
\tag{31}
$$

By the structure of $\mathbf{M}$ and by Lemmas 6 and 9, we obtain the following equalities:

$$
\begin{aligned}
\mathbf{M}(I_N\otimes A_i) &= (I_{N-k}\otimes A_i)\mathbf{M} = A_i^{\otimes}\mathbf{M},\\
\mathbf{M}(I_N\otimes B_i) &= (I_{N-k}\otimes B_i)\mathbf{M} = B_i^{\otimes}\mathbf{M},\\
\mathbf{M}(I_N\otimes C_i) &= (I_{N-k}\otimes C_i)\mathbf{M} = C_i^{\otimes}\mathbf{M},\\
\mathbf{M}(I_N\otimes D_i) &= (I_{N-k}\otimes D_i)\mathbf{M} = D_i^{\otimes}\mathbf{M},\\
\mathbf{M}(I_N\otimes E_i) &= (I_{N-k}\otimes E_i)\mathbf{M} = E_i^{\otimes}\mathbf{M},\\
\mathbf{M}(I_N\otimes F_i) &= (I_{N-k}\otimes F_i)\mathbf{M} = F_i^{\otimes}\mathbf{M},\\
\mathbf{M}\left(G_i^{(m)}\otimes\Gamma_{mi}\right) &= \left(\widetilde{M}\otimes I_N\right)\left(G_i^{(m)}\otimes\Gamma_{mi}\right)\\
&= \left(\widetilde{M}G_i^{(m)}\right)\otimes\Gamma_{mi} = \left(X_i^{(m)}\widetilde{M}\right)\otimes\Gamma_{mi}\\
&= \left(X_i^{(m)}\otimes\Gamma_{mi}\right)\left(\widetilde{M}\otimes I_N\right)\\
&= \overrightarrow{X_i^{(m)}}\mathbf{M}, \quad m=1,2,3,\\
\mathbf{M}(I_N\otimes L_i) &= (I_{N-k}\otimes L_i)\mathbf{M} = L_i^{\otimes}\mathbf{M}.
\end{aligned}
\tag{32}
$$

Taking Γ as its infinitesimal generator along the trajectory of (8), we obtain the following from Definition 3 and (30)–(3.1):

$$
\Gamma V(x(t),i,t) = \sum_{k=1}^{6}\Gamma V_k(x(t),i,t), \tag{33}
$$

$$
\begin{aligned}
\Gamma V_1(x(t),i,t) &= 2x^T(t)\mathbf{M}^T P_i\mathbf{M}\\
&\quad\times\left[\mathbb{A}_i x(t) + \mathbb{B}_i x(t-d_i(t))\right.\\
&\qquad + \mathbb{C}_i\dot{x}(t-\tau_i(t)) + \mathbb{D}_i F_1(x(t))\\
&\qquad + \mathbb{E}_i F_2(x(t-d_i(t)))\\
&\qquad \left.+\mathbb{F}_i F_3(\dot{x}(t-\tau_i(t))) + \mathbb{H}_i\omega(t)\right]\\
&\quad + \sum_{j\in S}\gamma_{ij}x^T(t)\mathbf{M}^T P_j\mathbf{M}x(t)\\
&= 2(\mathbf{M}x(t))^T P_i\left(A_i^{\otimes} + \overrightarrow{X_i^{(1)}}\right)(\mathbf{M}x(t))\\
&\quad + 2(\mathbf{M}x(t))^T P_i\left(B_i^{\otimes} + \overrightarrow{X_i^{(2)}}\right)(\mathbf{M}x(t))\\
&\quad + 2(\mathbf{M}x(t))^T P_i\left(C_i^{\otimes} + \overrightarrow{X_i^{(3)}}\right)(\mathbf{M}x(t))\\
&\quad + 2(\mathbf{M}x(t))^T P_i D_i^{\otimes}(\mathbf{M}F_1(x(t)))\\
&\quad + 2(\mathbf{M}x(t))^T P_i E_i^{\otimes}(\mathbf{M}F_2(x(t-d_i(t))))\\
&\quad + 2(\mathbf{M}x(t))^T P_i F_i^{\otimes}(\mathbf{M}F_3(\dot{x}(t-\tau_i(t))))\\
&\quad + 2(\mathbf{M}x(t))^T P_i\mathbb{H}_i\omega(t)\\
&\quad + (\mathbf{M}x(t))^T\left(\sum_{j\in S}\gamma_{ij}P_j\right)(\mathbf{M}x(t)),
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma V_2(x(t),i,t)\\
&\quad = (\mathbf{M}x(t))^T[Q_1+Q_4](\mathbf{M}x(t))\\
&\qquad + (\mathbf{M}x(t-\tau_{1i}))^T[Q_2-Q_1](\mathbf{M}x(t-\tau_{1i}))\\
&\qquad + (\mathbf{M}x(t-\tau_{mi}))^T[Q_3-Q_2](\mathbf{M}x(t-\tau_{mi}))\\
&\qquad - (\mathbf{M}x(t-\tau_{2i}))^T Q_3(\mathbf{M}x(t-\tau_{2i}))\\
&\qquad + (\mathbf{M}x(t-d_{1i}))^T[Q_5-Q_4](\mathbf{M}x(t-d_{1i}))\\
&\qquad + (\mathbf{M}x(t-d_{mi}))^T[Q_6-Q_5](\mathbf{M}x(t-d_{mi}))\\
&\qquad - (\mathbf{M}x(t-d_{2i}))^T Q_6(\mathbf{M}x(t-d_{2i})),
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma V_3(x(t),i,t)\\
&= (\mathbf{M}\dot{x}(t))^T[R_2+R_5](\mathbf{M}\dot{x}(t))\\
&\quad+(\mathbf{M}\dot{x}(t-\tau_{1i}))^T[R_1+R_3-R_2](\mathbf{M}\dot{x}(t-\tau_{1i}))\\
&\quad-(1-\dot{\tau}_i(t))(\mathbf{M}\dot{x}(t-\tau_i(t)))^T R_1(\mathbf{M}\dot{x}(t-\tau_i(t)))\\
&\quad-(\mathbf{M}\dot{x}(t-\tau_{2i}))^T R_4(\mathbf{M}\dot{x}(t-\tau_{2i}))\\
&\quad+(\mathbf{M}\dot{x}(t-\tau_{mi}))^T[R_4-R_3](\mathbf{M}\dot{x}(t-\tau_{mi}))\\
&\quad+(\mathbf{M}\dot{x}(t-d_{1i}))^T[R_6-R_5](\mathbf{M}\dot{x}(t-d_{1i}))\\
&\quad+(\mathbf{M}\dot{x}(t-d_{mi}))^T[R_7-R_6](\mathbf{M}\dot{x}(t-d_{mi}))\\
&\quad-(\mathbf{M}\dot{x}(t-d_{2i}))^T R_7(\mathbf{M}\dot{x}(t-d_{2i}))\\
&\quad+\sum_{j\in S}\gamma_{ij}\int_{t-\tau_j(t)}^{t-\tau_{1i}}\dot{x}^T(s)\mathbf{M}^T R_1\mathbf{M}\dot{x}(s)\,ds\\
&\le (\mathbf{M}\dot{x}(t))^T[R_2+R_5](\mathbf{M}\dot{x}(t))\\
&\quad+(\mathbf{M}\dot{x}(t-\tau_{1i}))^T[R_1+R_3-R_2](\mathbf{M}\dot{x}(t-\tau_{1i}))\\
&\quad-(1-\dot{\tau}_i(t))(\mathbf{M}\dot{x}(t-\tau_i(t)))^T R_1(\mathbf{M}\dot{x}(t-\tau_i(t)))\\
&\quad-(\mathbf{M}\dot{x}(t-\tau_{2i}))^T R_4(\mathbf{M}\dot{x}(t-\tau_{2i}))\\
&\quad+(\mathbf{M}\dot{x}(t-\tau_{mi}))^T[R_4-R_3](\mathbf{M}\dot{x}(t-\tau_{mi}))\\
&\quad+(\mathbf{M}\dot{x}(t-d_{1i}))^T[R_6-R_5](\mathbf{M}\dot{x}(t-d_{1i}))\\
&\quad+(\mathbf{M}\dot{x}(t-d_{mi}))^T[R_7-R_6](\mathbf{M}\dot{x}(t-d_{mi}))\\
&\quad-(\mathbf{M}\dot{x}(t-d_{2i}))^T R_7(\mathbf{M}\dot{x}(t-d_{2i}))\\
&\quad-\gamma_{ii}\int_{t-\overline{\tau}}^{t-\tau_{1i}}\dot{x}^T(s)\mathbf{M}^T R_1\mathbf{M}\dot{x}(s)\,ds\\
&\le (\mathbf{M}\dot{x}(t))^T[R_2+R_5](\mathbf{M}\dot{x}(t))\\
&\quad+(\mathbf{M}\dot{x}(t-\tau_{1i}))^T[R_1+R_3-R_2](\mathbf{M}\dot{x}(t-\tau_{1i}))\\
&\quad-(1-\nu_i)(\mathbf{M}\dot{x}(t-\tau_i(t)))^T R_1(\mathbf{M}\dot{x}(t-\tau_i(t)))\\
&\quad-(\mathbf{M}\dot{x}(t-\tau_{2i}))^T R_4(\mathbf{M}\dot{x}(t-\tau_{2i}))\\
&\quad+(\mathbf{M}\dot{x}(t-\tau_{mi}))^T[R_4-R_3](\mathbf{M}\dot{x}(t-\tau_{mi}))\\
&\quad+(\mathbf{M}\dot{x}(t-d_{1i}))^T[R_6-R_5](\mathbf{M}\dot{x}(t-d_{1i}))\\
&\quad+(\mathbf{M}\dot{x}(t-d_{mi}))^T[R_7-R_6](\mathbf{M}\dot{x}(t-d_{mi}))\\
&\quad-(\mathbf{M}\dot{x}(t-d_{2i}))^T R_7(\mathbf{M}\dot{x}(t-d_{2i}))\\
&\quad+\eta\int_{t-\overline{\tau}}^{t-\tau_{1i}}\dot{x}^T(s)\mathbf{M}^T R_1\mathbf{M}\dot{x}(s)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma V_4(x(t),i,t)\\
&= (\mathbf{M}x(t))^T\Big[\tau_{1i}^2T_1+d_{1i}^2T_4+(\tau_{mi}-\tau_{1i})^2T_2\\
&\qquad+(\tau_{2i}-\tau_{mi})^2T_3+(d_{mi}-d_{1i})^2T_5\\
&\qquad+(d_{2i}-d_{mi})^2T_6\Big](\mathbf{M}x(t))\\
&\quad-\int_{t-\tau_{1i}}^{t}\tau_{1i}x^T(s)\mathbf{M}^TT_1\mathbf{M}x(s)\,ds\\
&\quad-\int_{t-\tau_{mi}}^{t-\tau_{1i}}(\tau_{mi}-\tau_{1i})x^T(s)\mathbf{M}^TT_2\mathbf{M}x(s)\,ds\\
&\quad-\int_{t-\tau_{2i}}^{t-\tau_{mi}}(\tau_{2i}-\tau_{mi})x^T(s)\mathbf{M}^TT_3\mathbf{M}x(s)\,ds\\
&\quad-\int_{t-d_{1i}}^{t}d_{1i}x^T(s)\mathbf{M}^TT_4\mathbf{M}x(s)\,ds\\
&\quad-\int_{t-d_{mi}}^{t-d_{1i}}(d_{mi}-d_{1i})x^T(s)\mathbf{M}^TT_5\mathbf{M}x(s)\,ds\\
&\quad-\int_{t-d_{2i}}^{t-d_{mi}}(d_{2i}-d_{mi})x^T(s)\mathbf{M}^TT_6\mathbf{M}x(s)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma V_5(x(t),i,t)\\
&= (\mathbf{M}\dot{x}(t))^T\Big[\tau_{1i}^2U_1+d_{1i}^2U_4+(\tau_{mi}-\tau_{1i})^2U_2\\
&\qquad+(\tau_{2i}-\tau_{mi})^2U_3+(d_{mi}-d_{1i})^2U_5\\
&\qquad+(d_{2i}-d_{mi})^2U_6+\eta(\overline{\tau}-\tau_{1i})R_1\Big](\mathbf{M}\dot{x}(t))\\
&\quad-\int_{t-\tau_{1i}}^{t}\tau_{1i}\dot{x}^T(s)\mathbf{M}^TU_1\mathbf{M}\dot{x}(s)\,ds\\
&\quad-\int_{t-\tau_{mi}}^{t-\tau_{1i}}(\tau_{mi}-\tau_{1i})\dot{x}^T(s)\mathbf{M}^TU_2\mathbf{M}\dot{x}(s)\,ds\\
&\quad-\int_{t-\tau_{2i}}^{t-\tau_{mi}}(\tau_{2i}-\tau_{mi})\dot{x}^T(s)\mathbf{M}^TU_3\mathbf{M}\dot{x}(s)\,ds\\
&\quad-\int_{t-d_{1i}}^{t}d_{1i}\dot{x}^T(s)\mathbf{M}^TU_4\mathbf{M}\dot{x}(s)\,ds\\
&\quad-\int_{t-d_{mi}}^{t-d_{1i}}(d_{mi}-d_{1i})\dot{x}^T(s)\mathbf{M}^TU_5\mathbf{M}\dot{x}(s)\,ds\\
&\quad-\int_{t-d_{2i}}^{t-d_{mi}}(d_{2i}-d_{mi})\dot{x}^T(s)\mathbf{M}^TU_6\mathbf{M}\dot{x}(s)\,ds\\
&\quad-\eta\int_{t-\overline{\tau}}^{t-\tau_{1i}}\dot{x}^T(s)\mathbf{M}^TR_1\mathbf{M}\dot{x}(s)\,ds,
\end{aligned}
$$

$$\begin{aligned}
&\Gamma V_6(x(t),i,t)\\
&= (\mathbf{M}\dot{x}(t))^T\left[\frac{\tau_{1i}^4}{4}V_1+\frac{d_{1i}^4}{4}V_4+\frac{\left(\tau_{mi}^2-\tau_{1i}^2\right)^2}{4}V_2\right.\\
&\qquad+\frac{\left(\tau_{2i}^2-\tau_{mi}^2\right)^2}{4}V_3+\frac{\left(d_{mi}^2-d_{1i}^2\right)^2}{4}V_5\\
&\qquad\left.+\frac{\left(d_{2i}^2-d_{mi}^2\right)^2}{4}V_6\right](\mathbf{M}\dot{x}(t))\\
&-\int_{-\tau_{1i}}^{0}\int_{t+\theta}^{t}\frac{\tau_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^TV_1\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&-\int_{-\tau_{mi}}^{-\tau_{1i}}\int_{t+\theta}^{t}\frac{\tau_{mi}^2-\tau_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^TV_2\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&-\int_{-\tau_{2i}}^{-\tau_{mi}}\int_{t+\theta}^{t}\frac{\tau_{2i}^2-\tau_{mi}^2}{2}\dot{x}^T(s)\mathbf{M}^TV_3\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&-\int_{-d_{1i}}^{0}\int_{t+\theta}^{t}\frac{d_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^TV_4\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&-\int_{-d_{mi}}^{-d_{1i}}\int_{t+\theta}^{t}\frac{d_{mi}^2-d_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^TV_5\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&-\int_{-d_{2i}}^{-d_{mi}}\int_{t+\theta}^{t}\frac{d_{2i}^2-d_{mi}^2}{2}\dot{x}^T(s)\mathbf{M}^TV_6\mathbf{M}\dot{x}(s)\,ds\,d\theta.
\end{aligned}\tag{34}$$

Define

$$\begin{aligned}
\xi(t)=\operatorname{col}\Big\{&\mathbf{M}x(t)\quad \mathbf{M}x(t-\tau_i(t))\quad \mathbf{M}x(t-\tau_{1i})\\
&\mathbf{M}x(t-\tau_{mi})\quad \mathbf{M}x(t-\tau_{2i})\quad \mathbf{M}\dot{x}(t-\tau_i(t))\\
&\mathbf{M}\dot{x}(t-\tau_{1i})\quad \mathbf{M}\dot{x}(t-\tau_{mi})\quad \mathbf{M}\dot{x}(t-\tau_{2i})\\
&\mathbf{M}\int_{t-\tau_{1i}}^{t}x(s)\,ds\quad \mathbf{M}\int_{t-\tau_i(t)}^{t-\tau_{1i}}x(s)\,ds\\
&\mathbf{M}\int_{t-\tau_{2i}}^{t-\tau_i(t)}x(s)\,ds\quad \mathbf{M}\int_{t-\tau_{mi}}^{t-\tau_{1i}}x(s)\,ds\\
&\mathbf{M}\int_{t-\tau_{2i}}^{t-\tau_{mi}}x(s)\,ds\quad \mathbf{M}x(t-d_i(t))\quad \mathbf{M}x(t-d_{1i})\\
&\mathbf{M}x(t-d_{mi})\quad \mathbf{M}x(t-d_{2i})\quad \mathbf{M}\dot{x}(t-d_{1i})\\
&\mathbf{M}\dot{x}(t-d_{mi})\quad \mathbf{M}\dot{x}(t-d_{2i})\quad \mathbf{M}\int_{t-d_{1i}}^{t}x(s)\,ds\\
&\mathbf{M}\int_{t-d_i(t)}^{t-d_{1i}}x(s)\,ds\quad \mathbf{M}\int_{t-d_{2i}}^{t-d_i(t)}x(s)\,ds\\
&\mathbf{M}\int_{t-d_{mi}}^{t-d_{1i}}x(s)\,ds\quad \mathbf{M}\int_{t-d_{2i}}^{t-d_{mi}}x(s)\,ds\quad \mathbf{M}F_1(x(t))\\
&\mathbf{M}F_2(x(t-d_i(t)))\quad \mathbf{M}F_3(\dot{x}(t-\tau_i(t)))\quad \omega(t)\Big\}.
\end{aligned}\tag{35}$$

According to (7), we can obtain the following inequalities for any $\epsilon_1,\epsilon_2,\epsilon_3>0$:

$$\begin{aligned}
&\epsilon_1[\mathbf{M}F_1(x(t))]^T[\mathbf{M}F_1(x(t))]\\
&\quad=\epsilon_1\sum_{j=1}^{N-1}\left[f_1(x_j(t))-f_1(x_{j+1}(t))\right]^T\\
&\qquad\times\left[f_1(x_j(t))-f_1(x_{j+1}(t))\right]\\
&\quad\le\epsilon_1\sum_{j=1}^{N-1}\Big\{\left[f_1(x_j(t))-f_1(x_{j+1}(t))\right]^T W_2^{(1)}\\
&\qquad\times\left(x_j(t)-x_{j+1}(t)\right)+\left(x_j(t)-x_{j+1}(t)\right)^T\\
&\qquad\times W_1^{(1)^T}\left[f_1(x_j(t))-f_1(x_{j+1}(t))\right]\\
&\qquad-\left(x_j(t)-x_{j+1}(t)\right)^T W_1^{(1)^T}W_2^{(1)}\\
&\qquad\times\left(x_j(t)-x_{j+1}(t)\right)\Big\}\\
&\quad=\epsilon_1[\mathbf{M}F_1(x(t))]^TW_2^{(1)}[\mathbf{M}(x(t))]\\
&\qquad+\epsilon_1[\mathbf{M}(x(t))]^TW_1^{(1)^T}[\mathbf{M}F_1(x(t))]\\
&\qquad-\epsilon_1[\mathbf{M}(x(t))]^TW_1^{(1)^T}W_2^{(1)}[\mathbf{M}(x(t))],\\
&\epsilon_2[\mathbf{M}F_2(x(t-d_i(t)))]^T[\mathbf{M}F_2(x(t-d_i(t)))]\\
&\quad\le\epsilon_2[\mathbf{M}F_2(x(t-d_i(t)))]^TW_2^{(2)}[\mathbf{M}x(t-d_i(t))]\\
&\qquad+\epsilon_2[\mathbf{M}(x(t-d_i(t)))]^TW_1^{(2)^T}[\mathbf{M}F_2(x(t-d_i(t)))]\\
&\qquad-\epsilon_2[\mathbf{M}(x(t-d_i(t)))]^TW_1^{(2)^T}W_2^{(2)}\\
&\qquad\times[\mathbf{M}(x(t-d_i(t)))],\\
&\epsilon_3[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]^T[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]\\
&\quad\le\epsilon_3[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]^TW_2^{(3)}[\mathbf{M}(\dot{x}(t-\tau_i(t)))]\\
&\qquad+\epsilon_3[\mathbf{M}(\dot{x}(t-\tau_i(t)))]^TW_1^{(3)^T}[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]\\
&\qquad-\epsilon_3[\mathbf{M}(\dot{x}(t-\tau_i(t)))]^TW_1^{(3)^T}W_2^{(3)}[\mathbf{M}(\dot{x}(t-\tau_i(t)))].
\end{aligned}\tag{36}$$

From (33) and (36), we have

$$\begin{aligned}
&\Gamma V(x(t),i,t)\\
&\quad\le\sum_{k=1}^{6}\Gamma V_k(x(t),i,t)\\
&\qquad-2\epsilon_1[\mathbf{M}F_1(x(t))]^T[\mathbf{M}F_1(x(t))]\\
&\qquad+2\epsilon_1[\mathbf{M}F_1(x(t))]^TW_2^{(1)}(\mathbf{M}x(t))
\end{aligned}$$

$$
\begin{aligned}
&+2\epsilon_1(\mathbf{M}x(t))^T W_1^{(1)^T}[\mathbf{M}F_1(x(t))]\\
&-2\epsilon_1[\mathbf{M}x(t)]^T W_1^{(1)^T} W_2^{(1)}[\mathbf{M}x(t)]\\
&-2\epsilon_2[\mathbf{M}F_2(x(t-d_i(t)))]^T[\mathbf{M}F_2(x(t-d_i(t)))]\\
&+2\epsilon_2[\mathbf{M}F_2(x(t-d_i(t)))]^T W_2^{(2)}(\mathbf{M}x(t-d_i(t)))\\
&+2\epsilon_2(\mathbf{M}x(t-d_i(t)))^T W_1^{(2)^T}[\mathbf{M}F_2(x(t-d_i(t)))]\\
&-2\epsilon_2[\mathbf{M}x(t-d_i(t))]^T W_1^{(2)^T} W_2^{(2)}[\mathbf{M}x(t-d_i(t))]\\
&-2\epsilon_3[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]^T[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]\\
&+2\epsilon_3[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]^T W_2^{(3)}(\mathbf{M}\dot{x}(t-\tau_i(t)))\\
&+2\epsilon_3(\mathbf{M}\dot{x}(t-\tau_i(t)))^T W_1^{(3)^T}[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]\\
&-2\epsilon_3[\mathbf{M}\dot{x}(t-\tau_i(t))]^T W_1^{(3)^T} W_2^{(3)}[\mathbf{M}\dot{x}(t-\tau_i(t))].
\end{aligned}
\tag{37}
$$

Noticing (a) of Lemma 10, then

$$
\begin{aligned}
&-\int_{t-\tau_{1i}}^{t}\tau_{1i}x^T(s)\mathbf{M}^T T_1\mathbf{M}x(s)\,ds\\
&\quad\le -\left[\mathbf{M}\int_{t-\tau_{1i}}^{t}x(s)ds\right]^T T_1\left[\mathbf{M}\int_{t-\tau_{1i}}^{t}x(s)\,ds\right]\\
&\quad= -\xi^T(t)E_{10}T_1E_{10}^T\xi(t),\\
&-\int_{t-\tau_{1i}}^{t}\tau_{1i}\dot{x}^T(s)\mathbf{M}^T U_1\mathbf{M}\dot{x}(s)\,ds\\
&\quad\le -\xi^T(t)(E_1-E_3)U_1\left(E_1^T-E_3^T\right)\xi(t),\\
&-\int_{t-d_{1i}}^{t}d_{1i}x^T(s)\mathbf{M}^T T_4\mathbf{M}x(s)\,ds\le -\xi^T(t)E_{22}T_4E_{22}^T\xi(t),\\
&-\int_{t-d_{1i}}^{t}d_{1i}\dot{x}^T(s)\mathbf{M}^T U_4\mathbf{M}\dot{x}(s)\,ds\\
&\quad\le -\xi^T(t)(E_1-E_{16})U_4\left(E_1^T-E_{16}^T\right)\xi(t).
\end{aligned}
\tag{38}
$$

Noticing (b) of Lemma 10, then

$$
\begin{aligned}
&-\int_{-\tau_{1i}}^{0}\int_{t+\theta}^{t}\frac{\tau_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_1\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad\le -\xi^T(t)(\tau_{1i}E_1-E_{10})V_1\left(\tau_{1i}E_1^T-E_{10}^T\right)\xi(t),\\
&-\int_{-\tau_{mi}}^{-\tau_{1i}}\int_{t+\theta}^{t}\frac{\tau_{mi}^2-\tau_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_2\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad\le -\xi^T(t)\left[(\tau_{mi}-\tau_{1i})\ E_1-E_{13}\right]\\
&\qquad\times V_2\left[(\tau_{mi}-\tau_{1i})E_1^T-E_{13}^T\right]\xi(t),\\
&-\int_{-\tau_{2i}}^{-\tau_{mi}}\int_{t+\theta}^{t}\frac{\tau_{2i}^2-\tau_{mi}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_3\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad\le -\xi^T(t)\left[(\tau_{2i}-\tau_{mi})\ E_1-E_{14}\right]\\
&\qquad\times V_3\left[(\tau_{2i}-\tau_{mi})E_1^T-E_{14}^T\right]\xi(t),\\
&-\int_{-d_{1i}}^{0}\int_{t+\theta}^{t}\frac{d_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_4\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad\le -\xi^T(t)(d_{1i}E_1-E_{22})V_4\left(d_{1i}E_1^T-E_{22}^T\right)\xi(t),\\
&-\int_{-d_{mi}}^{-d_{1i}}\int_{t+\theta}^{t}\frac{d_{mi}^2-d_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_5\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad\le -\xi^T(t)\left[(d_{mi}-d_{1i})E_1-E_{25}\right]\\
&\qquad\times V_5\left[(d_{mi}-d_{1i})E_1^T-E_{25}^T\right]\xi(t),\\
&-\int_{-d_{2i}}^{-d_{mi}}\int_{t+\theta}^{t}\frac{d_{2i}^2-d_{mi}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_6\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad\le -\xi^T(t)\left[(d_{2i}-d_{mi})E_1-E_{26}\right]\\
&\qquad\times V_6\left[(d_{2i}-d_{mi})E_1^T-E_{26}^T\right]\xi(t).
\end{aligned}
\tag{39}
$$

If $\tau_i(t)\in[\tau_{1i},\tau_{mi}]$ and $d_i(t)\in[d_{1i},d_{mi}]$, let

$$
\lambda_{1i}(t)=\frac{\tau_i(t)-\tau_{1i}}{\tau_{mi}-\tau_{1i}},\qquad \kappa_{1i}(t)=\frac{d_i(t)-d_{1i}}{d_{mi}-d_{1i}}.
\tag{40}
$$

Then the following is held from (a) of Lemma 10:

$$
\begin{aligned}
&-\int_{t-\tau_{mi}}^{t-\tau_{1i}}(\tau_{mi}-\tau_{1i})x^T(s)\mathbf{M}^T T_2\mathbf{M}x(s)\,ds\\
&=-\left\{\int_{t-\tau_i(t)}^{t-\tau_{1i}}+\int_{t-\tau_{mi}}^{t-\tau_i(t)}\right\}(\tau_{mi}-\tau_{1i})x^T(s)\mathbf{M}^T T_2\mathbf{M}x(s)\,ds\\
&=-(\tau_{mi}-\tau_i(t))\int_{t-\tau_i(t)}^{t-\tau_{1i}}x^T(s)\mathbf{M}^T T_2\mathbf{M}x(s)\,ds\\
&\quad-(\tau_i(t)-\tau_{1i})\int_{t-\tau_i(t)}^{t-\tau_{1i}}x^T(s)\mathbf{M}^T T_2\mathbf{M}x(s)\,ds\\
&\quad-(\tau_{mi}-\tau_i(t))\int_{t-\tau_{mi}}^{t-\tau_i(t)}x^T(s)\mathbf{M}^T T_2\mathbf{M}x(s)\,ds\\
&\quad-(\tau_i(t)-\tau_{1i})\int_{t-\tau_{mi}}^{t-\tau_i(t)}x^T(s)\mathbf{M}^T T_2\mathbf{M}x(s)\,ds\\
&\le -\xi^T(t)E_{11}T_2E_{11}^T\xi(t)\\
&\quad-(1-\lambda_{1i}(t))\xi^T(t)E_{11}T_2E_{11}^T\xi(t)\\
&\quad-\xi^T(t)(E_{13}-E_{11})T_2\left(E_{13}^T-E_{11}^T\right)\xi(t)\\
&\quad-\lambda_{1i}(t)\xi^T(t)(E_{13}-E_{11})T_2\left(E_{13}^T-E_{11}^T\right)\xi(t).
\end{aligned}
\tag{41}
$$

Similarly,

$$\begin{aligned}
&-\int_{t-\tau_{mi}}^{t-\tau_{1i}}(\tau_{mi}-\tau_{1i})\dot{x}^T(s)\mathbf{M}^TU_2\mathbf{M}\dot{x}(s)\,ds\\
&=-\left\{\int_{t-\tau_i(t)}^{t-\tau_{1i}}+\int_{t-\tau_{mi}}^{t-\tau_i(t)}\right\}(\tau_{mi}-\tau_{1i})\dot{x}^T(s)\mathbf{M}^TT_2\mathbf{M}\dot{x}(s)\,ds\\
&\le-\xi^T(t)(E_3-E_2)U_2\left(E_3^T-E_2^T\right)\xi(t)\\
&\quad-(1-\lambda_{1i}(t))\xi^T(t)(E_3-E_2)U_2\left(E_3^T-E_2^T\right)\xi(t)\\
&\quad-\xi^T(t)(E_2-E_4)U_2\left(E_2^T-E_4^T\right)\xi(t)\\
&\quad-\lambda_{1i}(t)\xi^T(t)(E_2-E_4)U_2\left(E_2^T-E_4^T\right)\xi(t),\\
&-\int_{t-d_{mi}}^{t-d_{1i}}(d_{mi}-d_{1i})x^T(s)\mathbf{M}^TT_5\mathbf{M}x(s)\,ds\\
&=-\left\{\int_{t-d_i(t)}^{t-d_{1i}}+\int_{t-d_{mi}}^{t-d_i(t)}\right\}(d_{mi}-d_{1i})x^T(s)\mathbf{M}^TT_5\mathbf{M}x(s)\,ds\\
&\le-\xi^T(t)E_{23}T_5E_{23}^T\xi(t)-(1-\kappa_{1i}(t))\xi^T(t)E_{23}T_5E_{23}^T\xi(t)\\
&\quad-\xi^T(t)(E_{25}-E_{23})T_5\left(E_{25}^T-E_{23}^T\right)\xi(t)\\
&\quad-\kappa_{1i}(t)\xi^T(t)(E_{25}-E_{23})T_5\left(E_{25}^T-E_{23}^T\right)\xi(t),\\
&-\int_{t-d_{mi}}^{t-d_{1i}}(d_{mi}-d_{1i})\dot{x}^T(s)\mathbf{M}^TU_5\mathbf{M}\dot{x}(s)\,ds\\
&=-\left\{\int_{t-d_i(t)}^{t-d_{1i}}+\int_{t-d_{mi}}^{t-d_i(t)}\right\}(d_{mi}-d_{1i})\dot{x}^T(s)\mathbf{M}^TU_5\mathbf{M}\dot{x}(s)\,ds\\
&\le-\xi^T(t)(E_{16}-E_{15})U_5\left(E_{16}^T-E_{15}^T\right)\xi(t)\\
&\quad-(1-\kappa_{1i}(t))\xi^T(t)(E_{16}-E_{15})U_5\left(E_{16}^T-E_{15}^T\right)\xi(t)\\
&\quad-\xi^T(t)(E_{15}-E_{17})U_5\left(E_{15}^T-E_{17}^T\right)\xi(t)-\kappa_{1i}(t)\xi^T(t)\\
&\quad\times(E_{15}-E_{17})U_5\left(E_{15}^T-E_{17}^T\right)\xi(t).
\end{aligned}\tag{42}$$

Considering

$$\begin{aligned}
&-\int_{t-\tau_{2i}}^{t-\tau_{mi}}(\tau_{2i}-\tau_{mi})x^T(s)\mathbf{M}^TT_3\mathbf{M}x(s)\,ds,\\
&-\int_{t-\tau_{2i}}^{t-\tau_{mi}}(\tau_{2i}-\tau_{mi})\dot{x}^T(s)\mathbf{M}^TU_3\mathbf{M}\dot{x}(s)\,ds,\\
&-\int_{t-d_{2i}}^{t-d_{mi}}(d_{2i}-d_{mi})x^T(s)\mathbf{M}^TT_6\mathbf{M}x(s)\,ds,\\
&-\int_{t-d_{2i}}^{t-d_{mi}}(d_{2i}-d_{mi})\dot{x}^T(s)\mathbf{M}^TU_6\mathbf{M}\dot{x}(s)\,ds,
\end{aligned}\tag{43}$$

we have

$$\begin{aligned}
&-\int_{t-\tau_{2i}}^{t-\tau_{mi}}(\tau_{2i}-\tau_{mi})x^T(s)\mathbf{M}^TT_3\mathbf{M}x(s)\,ds\\
&\quad\le-\xi^T(t)E_{14}T_3E_{14}^T\xi(t),\\
&-\int_{t-\tau_{2i}}^{t-\tau_{mi}}(\tau_{2i}-\tau_{mi})\dot{x}^T(s)\mathbf{M}^TU_3\mathbf{M}\dot{x}(s)\,ds\\
&\quad\le-\xi^T(t)(E_4-E_5)U_3\left(E_4^T-E_5^T\right)\xi(t),\\
&-\int_{t-d_{2i}}^{t-d_{mi}}(d_{2i}-d_{mi})x^T(s)\mathbf{M}^TT_6\mathbf{M}x(s)\,ds\\
&\quad\le-\xi^T(t)E_{26}T_6E_{26}^T\xi(t),\\
&-\int_{t-d_{2i}}^{t-d_{mi}}(d_{2i}-d_{mi})\dot{x}^T(s)\mathbf{M}^TU_6\mathbf{M}\dot{x}(s)\,ds\\
&\quad\le-\xi^T(t)(E_{17}-E_{18})U_6\left(E_{17}^T-E_{18}^T\right)\xi(t).
\end{aligned}\tag{44}$$

In addition, according to (8), we know that $\mathbf{M}\dot{x}(t)=\Lambda\xi(t)$ and

$$[\mathbf{M}\dot{x}(t)]^TJ[\mathbf{M}\dot{x}(t)]=\xi^T(t)\Lambda^TJ\Lambda\xi(t),\tag{45}$$

where Λ and J have been defined in Theorem 12.

From (3.1) and (37)–(45), we obtain

$$\begin{aligned}
&\Gamma V(x(t),i,t)+\|\mathbf{M}z(t)\|^2-\delta^2\|\omega(t)\|^2\\
&\quad\le\sum_{k=1}^{6}\Gamma V_k(x(t),i,t)-2\epsilon_1[\mathbf{M}F_1(x(t))]^T[\mathbf{M}F_1(x(t))]\\
&\quad+2\epsilon_1[\mathbf{M}F_1(x(t))]^TW_2^{(1)}(\mathbf{M}x(t))+2\epsilon_1(\mathbf{M}x(t))^T\\
&\quad\times W_1^{(1)^T}[\mathbf{M}F_1(x(t))]-2\epsilon_1[\mathbf{M}x(t)]^TW_1^{(1)^T}W_2^{(1)}\\
&\quad\times[\mathbf{M}x(t)]-2\epsilon_2[\mathbf{M}F_2(x(t-d_i(t)))]^T\\
&\quad\times[\mathbf{M}F_2(x(t-d_i(t)))]\\
&\quad+2\epsilon_2[\mathbf{M}F_2(x(t-d_i(t)))]^TW_2^{(2)}(\mathbf{M}x(t-d_i(t)))\\
&\quad+2\epsilon_2(\mathbf{M}x(t-d_i(t)))^TW_1^{(2)^T}[\mathbf{M}F_2(x(t-d_i(t)))]\\
&\quad-2\epsilon_2[\mathbf{M}x(t-d_i(t))]^TW_1^{(2)^T}W_2^{(2)}[\mathbf{M}x(t-d_i(t))]\\
&\quad-2\epsilon_3[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]^T[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]\\
&\quad+2\epsilon_3[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]^TW_2^{(3)}(\mathbf{M}\dot{x}(t-\tau_i(t)))\\
&\quad+2\epsilon_3(\mathbf{M}\dot{x}(t-\tau_i(t)))^TW_1^{(3)^T}[\mathbf{M}F_3(\dot{x}(t-\tau_i(t)))]\\
&\quad-2\epsilon_3[\mathbf{M}\dot{x}(t-\tau_i(t))]^TW_1^{(3)^T}W_2^{(3)}[\mathbf{M}\dot{x}(t-\tau_i(t))]\\
&\quad+x^T(t)\mathbf{M}^TL_i^{\otimes^T}L_i^{\otimes}\mathbf{M}x(t)-\delta^2\omega^T(t)\omega(t)\\
&\le\xi^T(t)\left[\lambda_{1i(t)}\Omega_{i1}+(1-\lambda_{1i}(t))\Omega_{i2}+\frac{\Theta_{i0}}{2}\right]\xi(t)\\
&\quad+\xi^T(t)\left[\kappa_{1i(t)}\Pi_{i1}+(1-\kappa_{1i}(t))\Pi_{i2}+\frac{\Theta_{i0}}{2}\right]\xi(t).
\end{aligned}\tag{46}$$

For $\tau_i(t) \in [\tau_{mi}, \tau_{2i}]$ and $d_i(t) \in [d_{mi}, d_{2i}]$, let

$$\lambda_{2i}(t) = \frac{\tau_i(t) - \tau_{mi}}{\tau_{2i} - \tau_{mi}}, \qquad \kappa_{2i}(t) = \frac{d_i(t) - d_{mi}}{d_{2i} - d_{mi}}. \tag{47}$$

Then, following the above procedure, we can obtain

$$\begin{aligned} &\Gamma V(x(t), i, t) + \|\mathbf{M}z(t)\|^2 - \delta^2\|\omega(t)\|^2 \\ &\quad \le \xi^T(t)\left[\lambda_{2i}(t)\Omega_{i3} + (1-\lambda_{2i}(t))\Omega_{i4} + \frac{\Theta_{i0}}{2}\right]\xi(t) \\ &\qquad + \xi^T(t)\left[\kappa_{2i}(t)\Pi_{i3} + (1-\kappa_{2i}(t))\Pi_{i4} + \frac{\Theta_{i0}}{2}\right]\xi(t). \end{aligned} \tag{48}$$

For other situations, where $\tau_i(t) \in [\tau_{mi}, \tau_{2i}]$, $d_i(t) \in [d_{1i}, d_{mi}]$, and $\tau_i(t) \in [\tau_{1i}, \tau_{mi}]$, $d_i(t) \in [d_{mi}, d_{2i}]$, we derive (49) and (50), respectively, as

$$\begin{aligned} &\Gamma V(x(t), i, t) + \|\mathbf{M}z(t)\|^2 - \delta^2\|\omega(t)\|^2 \\ &\quad \le \xi^T(t)\left[\lambda_{2i}(t)\Omega_{i3} + (1-\lambda_{2i}(t))\Omega_{i4} + \frac{\Theta_{i0}}{2}\right]\xi(t) \\ &\qquad + \xi^T(t)\left[\kappa_{1i}(t)\Pi_{i1} + (1-\kappa_{1i}(t))\Pi_{i2} + \frac{\Theta_{i0}}{2}\right]\xi(t), \end{aligned} \tag{49}$$

$$\begin{aligned} &\Gamma V(x(t), i, t) + \|\mathbf{M}z(t)\|^2 - \delta^2\|\omega(t)\|^2 \\ &\quad \le \xi^T(t)\left[\lambda_{1i}(t)\Omega_{i1} + (1-\lambda_{1i}(t))\Omega_{i2} + \frac{\Theta_{i0}}{2}\right]\xi(t) \\ &\qquad + \xi^T(t)\left[\kappa_{2i}(t)\Pi_{i3} + (1-\kappa_{2i}(t))\Pi_{i4} + \frac{\Theta_{i0}}{2}\right]\xi(t). \end{aligned} \tag{50}$$

Therefore, with (46), (48), (49), and (50), by Lemma 11, the following inequality (51) is held for $\tau_i(t) \in [\tau_{1i}, \tau_{2i}]$ and $d_i(t) \in [d_{1i}, d_{2i}]$ if (25) is satisfied:

$$\Gamma V(x(t), i, t) + \|\mathbf{M}z(t)\|^2 - \delta^2\|\omega(t)\|^2 < 0. \tag{51}$$

If (26) is held, integrating the function in (51) from 0 to ∞, then we have

$$\begin{aligned} \int_0^\infty \|\mathbf{M}z(t)\|^2 dt &< \delta^2 \int_0^\infty \|\omega(t)\|^2 dt + V(0) \\ &\le \delta^2\left(\int_0^\infty \|\omega(t)\|^2 dt + x^T(0)Yx(0)\right). \end{aligned} \tag{52}$$

By Definition 8, the NCDNs (3) and (4) can reach H_∞ cluster synchronization with a disturbance attenuation δ. This completes the proof. □

Remark 13. It should be mentioned that the proposed Lyapunov functional contains some triple-integral terms. Compared with the existing ones, [39, 42] have shown that such a Lyapunov functional type is very effective in the reduction of conservatism. Besides, the information on the lower bound of the delay is sufficiently used by introducing the integral terms on $[t-\tau_i(t), t-\tau_{1i}]$, $[t-\tau_{1i}, t]$, $[t-\tau_{2i}, t-\tau_{mi}]$, $[t-\tau_{mi}, t-\tau_{1i}]$ and $[t-d_i(t), t-d_{1i}]$, $[t-d_{1i}, t]$, $[t-d_{2i}, t-d_{mi}]$, $[t-d_{mi}, t-d_{1i}]$.

Remark 14. H_∞ cluster synchronization of the neutral complex dynamical networks with Markovian switching is considered for the first time. The synchronization conditions are in the form of linear matrix inequalities (LMIs), which can be solved by utilizing the LMI toolbox in Matlab. The solvability of derived conditions depends not only on the attenuation level but also on the initial values of the complex networks.

In some special situations, the neutral delay may disappear and be regarded as $\tau_i(t) \equiv 0$, which can be described by the following equality and viewed as a general delayed complex dynamical network with Markovian switching:

$$\begin{aligned} \dot{x}(t) = {} & \mathbb{A}_i x(t) + \mathbb{B}_i x(t - d_i(t)) + \mathbb{D}_i F_1(x(t)) \\ & + \mathbb{E}_i F_2(x(t - d_i(t))) + \mathbb{H}_i \omega(t). \end{aligned} \tag{53}$$

The following corollary is therefore given to guarantee H_∞ cluster synchronization for this case.

Corollary 15. *Given the transition rate matrix Υ, the initial positive definite matrix $Y = Y^T > 0$, constant scalars d_{1i}, d_{2i}, and d_{mi} satisfying $d_{1i} < d_{mi} < d_{2i}$, the NCDN systems (53) and (4) with sector-bounded condition (7) are H_∞ cluster synchronization with a disturbance attenuation lever δ if there exist symmetric positive matrices $P_i > 0$, $(i \in S)$, $Q_j > 0$, $(j = 4, 5, 6)$, $R_k > 0$, $(k = 5, 6, 7)$, $T_l > 0$, $U_m > 0$, and $V_n > 0$, $(l, m, n = 4, 5, 6)$ for any scalars $\epsilon_1, \epsilon_2 > 0$ such that the following linear matrix inequalities hold:*

$$\begin{gathered} \overline{\Pi}_{i1} + \frac{1}{2}\overline{\Theta}_{i0} < 0, \qquad \overline{\Pi}_{i2} + \frac{1}{2}\overline{\Theta}_{i0} < 0, \\ \overline{\Pi}_{i3} + \frac{1}{2}\overline{\Theta}_{i0} < 0, \qquad \overline{\Pi}_{i4} + \frac{1}{2}\overline{\Theta}_{i0} < 0, \\ \overline{V}(0) < \delta^2 x^T(0) Y x(0), \end{gathered} \tag{54}$$

where

$$\begin{aligned} \overline{\Theta}_{i0} = {} & \sum_{m=1}^{16} \overline{E}_m \overline{\Phi}_m \overline{E}_m^T + \mathscr{L}(\overline{\Phi}) \\ & + \overline{\Lambda}^T \overline{J} \overline{\Lambda} - \left(\overline{E}_1 - \overline{E}_3\right) U_4 \left(\overline{E}_1^T - \overline{E}_3^T\right) \\ & - \left(d_{1i}\overline{E}_1 - \overline{E}_9\right) V_4 \left(d_{1i}\overline{E}_1^T - \overline{E}_9^T\right) \\ & - \left[(d_{mi} - d_{1i})\overline{E}_1 - \overline{E}_{12}\right] V_5 \left[(d_{mi} - d_{1i})\overline{E}_1^T - \overline{E}_{12}^T\right] \\ & - \left[(d_{2i} - d_{mi})\overline{E}_1 - \overline{E}_{13}\right] V_6 \left[(d_{2i} - d_{mi})\overline{E}_1^T - \overline{E}_{13}^T\right], \end{aligned} \tag{55}$$

where $\overline{E}_i$ $\{i = 1, 2, \ldots, 16\}$ are block entry matrices; that is,

$$\begin{aligned} \overline{E}_4^T &= [0 \;\; 0 \;\; 0 \;\; I \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0], \\ \overline{J} &= R_5 + d_{1i}^2 U_4 + (d_{mi} - d_{1i})^2 U_5 + (d_{2i} - d_{mi})^2 U_6 \\ &\quad + \frac{d_{1i}^4}{4} V_4 + \frac{\left(d_{mi}^2 - d_{1i}^2\right)^2}{4} V_5 + \frac{\left(d_{2i}^2 - d_{mi}^2\right)^2}{4} V_6, \end{aligned}$$

$$\begin{aligned}
\overline{\Lambda} &= \left(A_i^{\otimes} + \overrightarrow{X_i^{(1)}}\right)\overline{E}_1^T + \left(B_i^{\otimes} + \overrightarrow{X_i^{(2)}}\right)\overline{E}_2^T + D_i^{\otimes}\overline{E}_{14}^T \\
&\quad + E_i^{\otimes}\overline{E}_{15}^T + \mathbf{M}\mathbb{H}_i\overline{E}_{16}^T, \\
\overline{\Phi} &= \overline{E}_1\left(P_iD_i^{\otimes} + \epsilon_1 W_1^{(1)^T} + \epsilon_1 W_2^{(1)^T}\right)\overline{E}_{14}^T + \overline{E}_1P_iD_i^{\otimes}\overline{E}_{15}^T \\
&\quad + \overline{E}_1P_i\mathbb{H}_i\overline{E}_{16}^T + \overline{E}_2\left(\epsilon_2 W_1^{(2)^T} + \epsilon_2 W_2^{(2)^T}\right)\overline{E}_{15}^T, \\
\overline{\Phi}_1 &= \mathscr{L}\left[P_i\left(A_i^{\otimes} + B_i^{\otimes} + \overrightarrow{X_i^{(1)}} + \overrightarrow{X_i^{(2)}}\right) - \epsilon_1 W_1^{(1)^T}W_2^{(1)}\right] \\
&\quad + \sum_{j\in S}\gamma_{ij}P_j + Q_4 + d_{1i}^2T_4 + (d_{mi} - d_{1i})^2T_5 \\
&\quad + (d_{2i} - d_{mi})^2T_6 + L_i^{\otimes^T}L_i^{\otimes}, \\
\overline{\Phi}_2 &= -\epsilon_2 W_1^{(2)^T}W_2^{(2)}, \qquad \overline{\Phi}_3 = Q_5 - Q_4, \\
\overline{\Phi}_4 &= Q_6 - Q_5, \qquad \overline{\Phi}_5 = -Q_6, \\
\overline{\Phi}_6 &= R_6 - R_5, \qquad \overline{\Phi}_7 = R_7 - R_6, \qquad \overline{\Phi}_8 = -R_7, \\
\overline{\Phi}_{14} &= -2\epsilon_1 I, \qquad \overline{\Phi}_{15} = -2\epsilon_2 I, \\
\overline{\Phi}_{16} &= -\delta^2 I, \qquad \overline{\Phi}_m = 0, \quad (m = 9, 10, 11, 12, 13), \\
\overline{\Pi}_{i1} &= -\overline{E}_{10}T_5\overline{E}_{10}^T - 2\left(\overline{E}_{12} - \overline{E}_{10}\right)T_5\left(\overline{E}_{12}^T - \overline{E}_{10}^T\right) \\
&\quad - \left(\overline{E}_3 - \overline{E}_2\right)U_5\left(\overline{E}_3^T - \overline{E}_2^T\right) - 2\left(\overline{E}_2 - \overline{E}_4\right) \\
&\quad \times U_5\left(\overline{E}_2^T - \overline{E}_4^T\right) - \overline{E}_{13}T_6\overline{E}_{13}^T - \left(\overline{E}_4 - \overline{E}_5\right) \\
&\quad \times U_6\left(\overline{E}_4^T - \overline{E}_5^T\right), \\
\overline{\Pi}_{i2} &= -2\overline{E}_{10}T_5\overline{E}_{10}^T - \left(\overline{E}_{12} - \overline{E}_{10}\right)T_5\left(\overline{E}_{12}^T - \overline{E}_{10}^T\right) \\
&\quad - 2\left(\overline{E}_3 - \overline{E}_2\right)U_5\left(\overline{E}_3^T - \overline{E}_2^T\right) - \left(\overline{E}_2 - \overline{E}_4\right) \\
&\quad \times U_5\left(\overline{E}_2^T - \overline{E}_4^T\right) - \overline{E}_{13}T_6\overline{E}_{13}^T - \left(\overline{E}_4 - \overline{E}_5\right) \\
&\quad \times U_6\left(\overline{E}_4^T - \overline{E}_5^T\right), \\
\overline{\Pi}_{i3} &= -2\overline{E}_{11}T_6\overline{E}_{11}^T - \left(\overline{E}_{13} - \overline{E}_{11}\right)T_6\left(\overline{E}_{13}^T - \overline{E}_{11}^T\right) \\
&\quad - \left(\overline{E}_4 - \overline{E}_2\right)U_6\left(\overline{E}_4^T - \overline{E}_2^T\right) - 2\left(\overline{E}_2 - \overline{E}_5\right) \\
&\quad \times U_6\left(\overline{E}_2^T - \overline{E}_5^T\right) - \overline{E}_{12}T_5\overline{E}_{12}^T - \left(\overline{E}_3 - \overline{E}_4\right) \\
&\quad \times U_5\left(\overline{E}_3^T - \overline{E}_4^T\right), \\
\overline{\Pi}_{i4} &= -\overline{E}_{11}T_6\overline{E}_{11}^T - 2\left(\overline{E}_{13} - \overline{E}_{11}\right)T_6\left(\overline{E}_{13}^T - \overline{E}_{11}^T\right) \\
&\quad - 2\left(\overline{E}_4 - \overline{E}_2\right)U_6\left(\overline{E}_4^T - \overline{E}_2^T\right) - \left(\overline{E}_2 - \overline{E}_5\right) \\
&\quad \times U_6\left(\overline{E}_2^T - \overline{E}_5^T\right) - \overline{E}_{12}T_5\overline{E}_{12}^T - \left(\overline{E}_3 - \overline{E}_4\right) \\
&\quad \times U_5\left(\overline{E}_3^T - \overline{E}_4^T\right).
\end{aligned} \tag{56}$$

Other notations are the same as those in Theorem 12.

Proof. Since $\tau_i(t) \equiv 0$, we choose the Lyapunov functional as follows:

$$\overline{V}(x(t), i, t) = V_1(x(t), i, t) + \sum_{k=2}^{6}\overline{V}_k(x(t), i, t), \tag{57}$$

where

$$\begin{aligned}
\overline{V}_2(x(t), i, t) &= \int_{t-d_{1i}}^{t} x^T(s)\mathbf{M}^TQ_4\mathbf{M}x(s)\,ds \\
&\quad + \int_{t-d_{mi}}^{t-d_{1i}} x^T(s)\mathbf{M}^TQ_5\mathbf{M}x(s)\,ds \\
&\quad + \int_{t-d_{2i}}^{t-d_{mi}} x^T(s)\mathbf{M}^TQ_6\mathbf{M}x(s)\,ds, \\
\overline{V}_3(x(t), i, t) &= \int_{t-d_{1i}}^{t} \dot{x}^T(s)\mathbf{M}^TR_5\mathbf{M}\dot{x}(s)\,ds \\
&\quad + \int_{t-d_{mi}}^{t-d_{1i}} \dot{x}^T(s)\mathbf{M}^TR_6\mathbf{M}\dot{x}(s)\,ds \\
&\quad + \int_{t-d_{2i}}^{t-d_{mi}} \dot{x}^T(s)\mathbf{M}^TR_7\mathbf{M}\dot{x}(s)\,ds, \\
&\overline{V}_4(x(t), i, t) \\
&= \int_{-d_{1i}}^{0}\int_{t+\theta}^{t} d_{1i}x^T(s)\mathbf{M}^TT_4\mathbf{M}x(s)\,ds\,d\theta \\
&\quad + \int_{-d_{mi}}^{-d_{1i}}\int_{t+\theta}^{t} (d_{mi} - d_{1i})x^T(s)\mathbf{M}^T \\
&\qquad \times T_5\mathbf{M}x(s)\,ds\,d\theta \\
&\quad + \int_{-d_{2i}}^{-d_{mi}}\int_{t+\theta}^{t} (d_{2i} - d_{mi})x^T(s)\mathbf{M}^T \\
&\qquad \times T_6\mathbf{M}x(s)\,ds\,d\theta,
\end{aligned}$$

$$\begin{aligned}
&\overline{V}_5(x(t),i,t)\\
&= \int_{-d_{1i}}^{0}\int_{t+\theta}^{t} d_{1i}\dot{x}^T(s)\mathbf{M}^T U_4 \mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-d_{mi}}^{-d_{1i}}\int_{t+\theta}^{t} (d_{mi}-d_{1i})\dot{x}^T(s)\mathbf{M}^T\\
&\qquad\qquad \times U_5\mathbf{M}\dot{x}(s)\,ds\,d\theta\\
&\quad + \int_{-d_{2i}}^{-d_{mi}}\int_{t+\theta}^{t} (d_{2i}-d_{mi})\dot{x}^T(s)\mathbf{M}^T\\
&\qquad\qquad \times U_6\mathbf{M}\dot{x}(s)\,ds\,d\theta,\\
&\overline{V}_6(x(t),i,t)\\
&= \int_{-d_{1i}}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \frac{d_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T V_4\mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-d_{mi}}^{-d_{1i}}\int_{\theta}^{0}\int_{t+\lambda}^{t} \frac{d_{mi}^2-d_{1i}^2}{2}\dot{x}^T(s)\mathbf{M}^T\\
&\qquad\qquad \times V_5\mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta\\
&\quad + \int_{-d_{2i}}^{-d_{mi}}\int_{\theta}^{0}\int_{t+\lambda}^{t} \frac{d_{2i}^2-d_{mi}^2}{2}\dot{x}^T(s)\mathbf{M}^T\\
&\qquad\qquad \times V_6\mathbf{M}\dot{x}(s)\,ds\,d\lambda\,d\theta.
\end{aligned} \tag{58}$$

And we define

$$\begin{aligned}
\overline{\xi}(t) = \operatorname{col}\Big\{&\mathbf{M}x(t)\quad \mathbf{M}x(t-d_i(t))\quad \mathbf{M}x(t-d_{1i})\\
&\mathbf{M}x(t-d_{mi})\quad \mathbf{M}x(t-d_{2i})\quad \mathbf{M}\dot{x}(t-d_{1i})\\
&\mathbf{M}\dot{x}(t-d_{mi})\quad \mathbf{M}\dot{x}(t-d_{2i})\quad \mathbf{M}\int_{t-d_{1i}}^{t} x(s)\,ds\\
&\mathbf{M}\int_{t-d_i(t)}^{t-d_{1i}} x(s)\,ds\quad \mathbf{M}\int_{t-d_{2i}}^{t-d_i(t)} x(s)\,ds\\
&\mathbf{M}\int_{t-d_{mi}}^{t-d_{1i}} x(s)\,ds\quad \mathbf{M}\int_{t-d_{2i}}^{t-d_{mi}} x(s)\,ds\\
&\mathbf{M}F_1(x(t))\quad \mathbf{M}F_2(x(t-d_i(t)))\quad \omega(t)\Big\}.
\end{aligned} \tag{59}$$

Then we follow a similar line as in proof of Theorem 12 and obtain the result. □

4. Numerical Examples

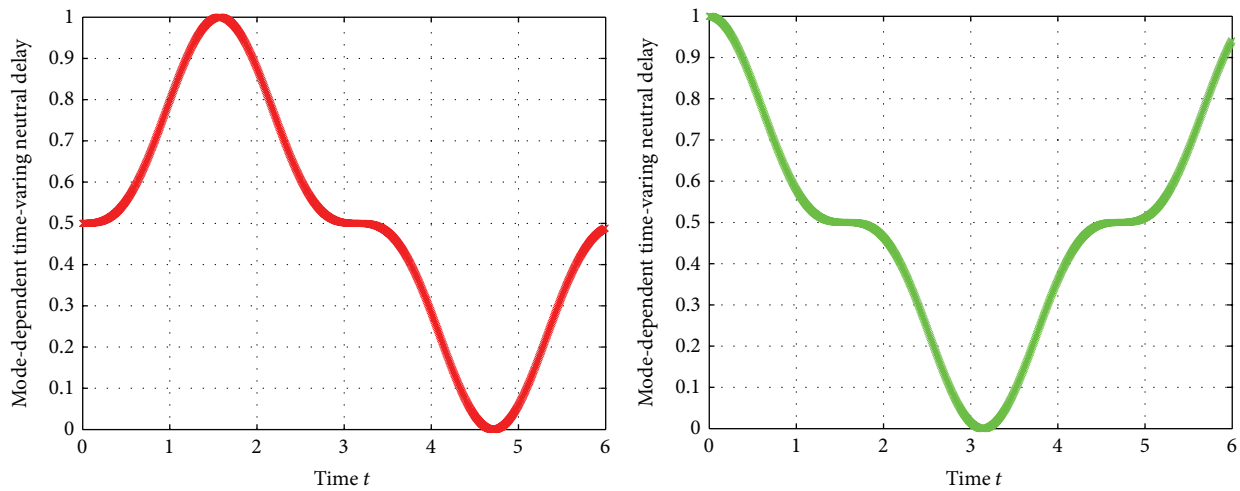
In this section, numerical examples are presented to demonstrate the effectiveness of the developed design on H_∞ cluster synchronization.

Example 1. A four-node NCDN (3) and (4) with Markovian switching between two modes is taken into consideration; that is, $N = 4$ and $M = 2$. The parametric matrices of the NCDN are given as follows:

$$\begin{gathered}
A_1 = \begin{bmatrix} -0.40 & -0.15\\ 0.10 & -0.60\end{bmatrix},\qquad A_2 = \begin{bmatrix} -0.30 & 0.09\\ 0.20 & -0.40\end{bmatrix},\\
B_1 = \begin{bmatrix} 0.20 & -0.15\\ 0.50 & -0.50\end{bmatrix},\qquad B_2 = \begin{bmatrix} 0.31 & 0.23\\ -0.12 & 0.17\end{bmatrix},\\
C_1 = \begin{bmatrix} 0.28 & 0.02\\ -0.06 & 0.11\end{bmatrix},\qquad C_2 = \begin{bmatrix} 0.51 & 0.24\\ 0.02 & -0.44\end{bmatrix},\\
D_1 = \begin{bmatrix} 0.20 & 0\\ 0 & -0.15\end{bmatrix},\qquad D_2 = \begin{bmatrix} 0.30 & 0\\ -0.10 & 0.23\end{bmatrix},\\
E_i = F_i = 0,\quad (i = 1,2),\\
G_1^{(1)} = \begin{bmatrix} -0.3 & 0.1 & 0.1 & 0.1\\ 0.1 & -0.3 & 0.1 & 0.1\\ 0.1 & 0.1 & -0.3 & 0.1\\ 0.1 & 0.1 & 0.1 & -0.3\end{bmatrix},\\
G_2^{(1)} = \begin{bmatrix} -0.1 & 0 & 0 & 0.1\\ 0.1 & -0.1 & 0 & 0\\ 0.1 & 0 & -0.1 & 0\\ 0 & 0 & 0.1 & -0.1\end{bmatrix},\\
G_1^{(2)} = \begin{bmatrix} -0.2 & 0.1 & 0.1 & 0\\ 0 & -0.2 & 0.1 & 0.1\\ 0.1 & 0 & -0.2 & 0.1\\ 0.1 & 0 & 0.1 & -0.2\end{bmatrix},\\
G_2^{(2)} = \begin{bmatrix} -0.2 & 0.1 & 0.1 & 0\\ 0 & 0 & 0 & 0\\ 0.1 & 0.1 & -0.2 & 0\\ 0.1 & 0.1 & 0 & -0.2\end{bmatrix},\\
G_1^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0.1 & -0.3 & 0.1 & 0.1\\ 0 & 0.1 & -0.3 & 0.1\\ 0.1 & 0.1 & 0.1 & -0.3\end{bmatrix},\\
G_2^{(3)} = \begin{bmatrix} -0.2 & 0 & 0.1 & 0.1\\ 0.1 & -0.2 & 0.1 & 0.1\\ 0 & 0 & 0 & 0\\ 0.1 & 0 & 0.1 & -0.2\end{bmatrix},\\
\Gamma_{1i} = \Gamma_{2i} = \Gamma_{3i} = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix},\qquad L_i = \begin{bmatrix} 1 & 1\\ 1 & 0\end{bmatrix},\\
i \in S = \{1,2\},\\
H_{k1} = \begin{bmatrix} 0\\ 1\end{bmatrix},\qquad H_{k2} = \begin{bmatrix} 1\\ 0\end{bmatrix}\\
(k = 1,2,3,4).
\end{gathered} \tag{60}$$

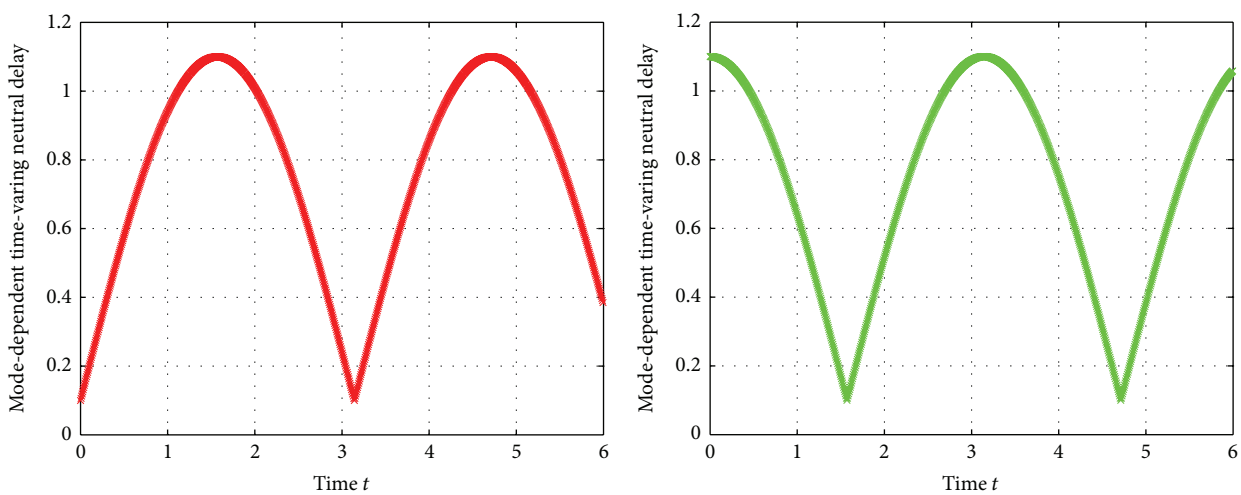
The transition rate matrix is considered as follows:

$$\Upsilon = \begin{bmatrix} -1 & 1\\ 2 & -2\end{bmatrix}. \tag{61}$$



(a) (b)

FIGURE 1: Mode-dependent time-varying neutral delays $\tau_i(t)$ at mode 1 and mode 2.



(a) (b)

FIGURE 2: Mode-dependent time-varying retarded delays $d_i(t)$ at mode 1 and mode 2.

Furthermore, as a result of $E_i = F_i = 0$, only the nonlinear function $f_1(x_k(t))$ is effective and given as

$$f_1(x_k(t)) = [0.5x_{k1}(t) - \tanh(0.2x_{k1}(t)) + 0.2x_{k2}(t) \quad 0.95x_{k2}(t) - \tanh(0.75x_{k2}(t))]^T. \tag{62}$$

Then, it is easy to verify that

$$W_1^{(1)} = \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0.3 \end{bmatrix}, \qquad W_2^{(1)} = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 0.95 \end{bmatrix}. \tag{63}$$

The interval mode-dependent time-varying neutral delays and discrete delays are, respectively, assumed to be

$$\begin{aligned} \tau_1(t) &= 0.5\left(1 + \sin^3(t)\right), \qquad \tau_2(t) = 0.5\left(1 + \cos^3(t)\right), \\ d_1(t) &= 0.1 + |\sin t|, \qquad d_2(t) = 0.1 + |\cos t|. \end{aligned} \tag{64}$$

They are governed by the Markov process $\{r(t), t \geq 0\}$ and shown in Figures 1 and 2. It can be readily obtained that

$$\begin{aligned} &\tau_{11} = 0, \qquad \tau_{21} = 1; \qquad \tau_{12} = 0, \\ &\tau_{22} = 1; \qquad \nu_1 = \nu_2 = \frac{\sqrt{3}}{3}, \\ &d_{11} = 0.1, \qquad d_{21} = 1.1; \\ &d_{12} = 0.1, \qquad d_{22} = 1.1. \end{aligned} \tag{65}$$

H_∞ cluster synchronization of this NCDN based on the above criterion is tested. Choose $\tau_{m1} = 0.2$, $\tau_{m2} = 0.3$, $d_{m1} = 0.4$, $d_{m2} = 0.5$, and the initial conditions

$$x_1(s) = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}, \qquad x_2(s) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix},$$
$$x_3(s) = \begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}, \qquad x_4(s) = \begin{bmatrix} 0.3 \\ -0.2 \end{bmatrix}, \tag{66}$$
$$s \in [-\varsigma, 0].$$

Let the disturbance attenuation level $\delta = 0.5$, and let the initial positive definite matrix $Y = 3I_8$. With Theorem 12, by using the Matlab LMI Toolbox, a group of matrices as a feasible solution can be obtained in the following (for simplicity, we only list the matrices for P_i and Q_j, $i \in S$, $j = 1, 2, \ldots, 6$):

$$P_1 = \begin{bmatrix} 1.7802 & 0.0659 & -0.0047 & 0.0028 \\ * & 1.0304 & 0.0012 & -0.0015 \\ * & * & 1.8546 & 0.0326 \\ * & * & * & 1.1325 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 1.6372 & 0.1644 & -0.0042 & 0.0040 \\ * & 1.4526 & 0.0033 & -0.0025 \\ * & * & 1.3748 & 0.0727 \\ * & * & * & 1.2369 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 2.3589 & 0.0467 & -0.0019 & 0.0016 \\ * & 3.1324 & 0.0016 & -0.0015 \\ * & * & 2.1046 & 0.4326 \\ * & * & * & 3.1433 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 3.0811 & 0.0259 & -0.0034 & 0.0027 \\ * & 3.3245 & 0.0029 & -0.0038 \\ * & * & 3.6435 & 0.0037 \\ * & * & * & 3.1046 \end{bmatrix},$$
$$Q_3 = \begin{bmatrix} 2.3042 & 0.1654 & -0.0003 & 0.0002 \\ * & 1.6345 & 0.0018 & -0.0014 \\ * & * & 1.8673 & 0.0756 \\ * & * & * & 1.0564 \end{bmatrix}, \tag{67}$$
$$Q_4 = \begin{bmatrix} 2.3632 & 0.0735 & -0.0011 & 0.0007 \\ * & 2.0411 & 0.0134 & -0.0001 \\ * & * & 1.7745 & 0.0542 \\ * & * & * & 1.0643 \end{bmatrix},$$
$$Q_5 = \begin{bmatrix} 3.1822 & -0.0453 & -0.0003 & -0.0005 \\ * & 3.3314 & 0 & 0 \\ * & * & 3.2446 & -0.0443 \\ * & * & * & 3.0418 \end{bmatrix},$$
$$Q_6 = \begin{bmatrix} 2.6433 & -0.0059 & -0.0050 & 0.0042 \\ * & 2.3074 & 0.0014 & -0.0005 \\ * & * & 2.0435 & 0.0926 \\ * & * & * & 1.8663 \end{bmatrix}.$$

It can be concluded that this neutral complex dynamical network (NCDN) has achieved H_∞ cluster synchronization, which illustrates the effectiveness of Theorem 12.

Example 2. Particularly, consider $\tau_i(t) \equiv 0$ in Example 1 and other elements are identical with Example 1. With Corollary 15, by utilizing Matlab LMI Toolbox, the LMIs (54) can be solved. Then a group of matrices as a feasible solution can be obtained as follows (for simplicity, we only list the matrices for P_i and Q_j, $i \in S$, $j = 4, 5, 6$):

$$P_1 = \begin{bmatrix} 1.5433 & 0.0049 & -0.0006 & 0.0003 \\ * & 1.0241 & 0.0018 & -0.0017 \\ * & * & 1.0327 & 0.0034 \\ * & * & * & 1.0065 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 1.5638 & 0.1536 & -0.0032 & 0.0028 \\ * & 1.3674 & 0.0026 & -0.0011 \\ * & * & 1.2655 & 0.0424 \\ * & * & * & 1.0258 \end{bmatrix},$$
$$Q_4 = \begin{bmatrix} 2.1844 & 0.0632 & -0.0009 & 0.0006 \\ * & 2.0087 & 0.0136 & -0.0001 \\ * & * & 1.7549 & 0.0466 \\ * & * & * & 1.0557 \end{bmatrix}, \tag{68}$$
$$Q_5 = \begin{bmatrix} 3.1756 & -0.0346 & -0.0003 & -0.0004 \\ * & 3.3267 & 0 & 0 \\ * & * & 3.2338 & -0.0365 \\ * & * & * & 3.0344 \end{bmatrix},$$
$$Q_6 = \begin{bmatrix} 2.6368 & -0.0047 & -0.0028 & 0.0035 \\ * & 2.2866 & 0.0006 & -0.0003 \\ * & * & 2.0337 & 0.0677 \\ * & * & * & 1.8359 \end{bmatrix}.$$

It also can be proved that the complex dynamical network (CDN) has achieved H_∞ cluster synchronization, which verifies the effectiveness of Corollary 15.

5. Conclusions

In this paper, H_∞ cluster synchronization of neutral complex dynamical networks with Markovian switching is considered for the first time. By interval mode-dependent delays dividing, a new augmented Lyapunov functional containing some triple-integral terms is constructed to reduce conservativeness. Then the delay-range-dependent H_∞ cluster synchronization criteria are obtained by the Lyapunov stability theory, integral matrix inequalities, and convex combination. Finally, numerical examples are given to illustrate the feasibility and effectiveness of the proposed result.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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