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Research Article

Littlewood-Paley Operators on Morrey Spaces with Variable Exponent

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By applying the vector-valued inequalities for the Littlewood-Paley operators and their commutators on Lebesgue spaces with variable exponent, the boundedness of the Littlewood-Paley operators, including the Lusin area integrals, the Littlewood-Paley g-functions and g_μ^*-functions, and their commutators generated by BMO functions, is obtained on the Morrey spaces with variable exponent.

1. Introduction and Main Results

Let $\psi \in L^1(\mathbb{R}^n)$ and satisfy the following:

(i) $\int_{\mathbb{R}^n} \psi(x)dx = 0$;

(ii) $|\psi(x)| \leqslant C(1+|x|)^{-n-\varepsilon}$;

(iii) $|\psi(x+y)-\psi(x)| \leqslant C|y|^\gamma(1+|x|)^{-n-\gamma-\varepsilon}$, $|x| \geqslant 2|y|$,

where C, ε, γ are all positive constants. Denote $\psi_t(x) = t^{-n}\psi(x/t)$ with $t > 0$ and $x \in \mathbb{R}^n$. Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Lusin area integral of f is defined by

$$S_{\psi,a}(f)(x) = \left(\int_{\Gamma_a(x)} |\psi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}, \tag{1}$$

where $\Gamma_a(x)$ denote the usual cone of aperture one

$$\Gamma_a(x) = \left\{(t,y) \in \mathbb{R}^{n+1}_+ : |y-x| < at,\ a \geq 0\right\}. \tag{2}$$

As $a = 1$, we denote $S_{\psi,a}(f)$ as $S_\psi(f)$.

Now let us turn to the introduction of the other two Littlewood-Paley operators. It is well known that the Littlewood-Paley operators include also the Littlewood-Paley g-functions and the Littlewood-Paley g_μ^*-functions besides the Lusin area integrals. The Littlewood-Paley g-functions, which can be viewed as a "zero-aperture" version of S_ψ, and g_μ^*-functions, which can be viewed as an "infinite-aperture" version of S_ψ, are, respectively, defined by

$$g(f)(x) = \left(\int_0^\infty |\psi_t * f(y)|^2 \frac{dt}{t}\right)^{1/2}, \tag{3}$$

$$g_\mu^*(f)(x) = \left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\mu n} |\psi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}, \quad \mu > 0. \tag{4}$$

If we take ψ to be the Poisson kernel, then the functions defined above are the classical Littlewood-Paley operators.

Letting $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $m \geq 1$, the corresponding m-order commutators of Littlewood-Paley operators above generated by a function b are defined by

$$\left[b^m, g_\psi\right](f)(x) = \left(\int_0^\infty \left|\int_{\mathbb{R}^n} [b(x)-b(y)]^m \psi_t(x-y) f(y)\,dy\right|^2 \frac{dt}{t}\right)^{1/2}, \tag{5}$$

$$\left[b^m, S_{\psi,a}\right](f)(x) = \left(\int_{\Gamma_a(x)} \left|\int_{\mathbb{R}^n} [b(x)-b(z)]^m \psi_t(y-z) f(z)\, dz\right|^2 \frac{dy\, dt}{t^{n+1}}\right)^{1/2}, \tag{6}$$

$$\left[b^m, g_\mu^*\right](f)(x) = \left(\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\mu n} \times \left|\int_{\mathbb{R}^n} [b(x)-b(z)]^m \psi_t(y-z) f(z)\, dz\right|^2 \frac{dy\, dt}{t^{n+1}}\right)^{1/2}, \tag{7}$$

where $\mu > 0$.

The Littlewood-Paley operators are a class of important integral operators. Due to the fact that they play very important roles in harmonic analysis, PDE, and the other fields (see [1–3]), people pay much more attention to this class of operators. In 1995, Lu and Yang investigated the behavior of Littlewood-Paley operators in the space $\mathrm{CBMO}_p(\mathbb{R}^n)$ in [4]. In 2005, Zhang and Liu proved the commutator $[b, g_\psi]$ is bounded on $L^p(\omega)$ in [5]. In 2009, Xue and Ding gave the weighted estimate for Littlewood-Paley operators and their commutators (see [6]). There are some other results about Littlewood-Paley operators in [7–9] and so forth.

On the other hand, Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ become one class of important research subject in analysis filed due to the fundamental paper [10] by Kováčik and Rákosník. In the past twenty years, the theory of these spaces has made progress rapidly, and the study of which has many applications in fluid dynamics, elasticity, calculus of variations, and differential equations with nonstandard growth conditions (see [11–15]). In [16], Cruz-Uribe et al. stated that the extrapolation theorem leads the boundedness of some classical operators including the commutator on $L^{p(\cdot)}(\mathbb{R}^n)$. Karlovich and Lerner also independently obtained the boundedness of the singular integrals commutator on Lebesgue spaces with variable exponent in [17]. In 2009 and 2010, Izuki considered the boundedness of vector-valued sublinear operators and fractional integrals on Herz-Morrey spaces with variable exponent in [18, 19], respectively. In 2013, Ho in [20] introduced a class of Morrey spaces with variable exponent $\mathscr{M}_{p(\cdot),u}$ and studied the boundedness of the fractional integral operators on $\mathscr{M}_{p(\cdot),u}$.

Inspired by the results mentioned previously, in this paper we will consider the vector-valued inequalities of the Littlewood-Paley operators and their m-order commutators on Morrey spaces with variable exponent. Before stating our main results, we need to recall some relevant definitions and notations.

Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with measure $|E| > 0$.

Definition 1 (see [10]). Let $p(\cdot) : E \to [1, \infty)$ be a measurable function.

The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \left\{ f \text{ is measurable: } \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx < \infty \text{ for some constant } \eta > 0 \right\}. \tag{8}$$

The space $L_{\mathrm{loc}}^{p(\cdot)}(E)$ is defined by

$$L_{\mathrm{loc}}^{p(\cdot)}(E) = \left\{ f \text{ is measurable: } f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset E \right\}. \tag{9}$$

The Lebesgue space $L^{p(\cdot)}(E)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{ \eta > 0 : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx \le 1 \right\}. \tag{10}$$

Remark 2. (1) Note that if the function $p(x) = p_0$ is a constant function, then $L^{p(\cdot)}(\mathbb{R}^n)$ equals $L^{p_0}(\mathbb{R}^n)$. This implies that the Lebesgue spaces with variable exponent generalize the usual Lebesgue spaces. And they have many properties in common with the usual Lebesgue spaces.

(2) Denote $p_- := \operatorname{ess\,inf}\{p(x) : x \in E\}$, $p_+ := \operatorname{ess\,sup}\{p(x) : x \in E\}$. Then $\mathscr{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

(3) The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy. \tag{11}$$

Denote $\mathscr{B}(E)$ to be the set of all functions $p(\cdot) \in \mathscr{P}(E)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(E)$.

(4) Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Denote $\kappa_{p(\cdot)} = \sup\{q > 1 : p(\cdot)/q \in \mathscr{B}(\mathbb{R}^n)\}$ and $e_{p(\cdot)}$ is the conjugate exponent of $\kappa_{p'(\cdot)}$ (see [20]).

Definition 3 (see [20]). Let $p(x) \in L^\infty(\mathbb{R}^n)$, $1 < p(x) < \infty$. If there exists a constant $C > 0$ such that, for any $x \in \mathbb{R}^n$ and $r > 0$, Lebesgue measurable function $u(x, r) : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ satisfying

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u\left(x, 2^{j+1} r\right) \le C u(x, r), \tag{12}$$

then one says u is a Morrey weight function for $L^{p(\cdot)}(\mathbb{R}^n)$. One denotes the class of Morrey weight functions by $\mathbb{W}_{p(\cdot)}$.

Definition 4 (see [20]). Let $p(x) \in \mathscr{B}(\mathbb{R}^n)$, $u(x,r) \in \mathbb{W}_{p(\cdot)}$. Then the Morrey spaces with variable exponent $\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)$ are defined by

$$\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n) = \left\{ f \text{ is measurable: } \|f\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty \right\}, \quad (13)$$

where

$$\|f\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, R>0} \frac{1}{u(x,R)} \|f\chi_{B(x,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \quad (14)$$

Remark 5. (1) If $u(x,r) \equiv 1$, then $\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)$ is the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$.

(2) Notice that if $p(x) \equiv p$, $1 < p < \infty$, is a constant function, then formula (12) can be rewritten as an integral in form. To be precise, formula (12) can be rewritten in the following form (see [20]):

$$\int_r^\infty \frac{u(x,t)}{t^{n/p+1}} dt \le C \frac{u(x,r)}{r^{n/p}}, \quad r > 0, \ \forall x \in \mathbb{R}^n. \quad (15)$$

Let $0 < \alpha < n$. By the the conditions of Morrey weight functions mentioned in [21]

$$\int_r^\infty \frac{u^p(x,t)}{t^{n-\alpha/p+1}} dt \le C \frac{u^p(x,r)}{r^{n-\alpha/p}}, \quad r > 0, \ \forall x \in \mathbb{R}^n \quad (16)$$

and Hölder's inequality, via simple calculation, we have

$$\begin{aligned} &\int_r^\infty \frac{u(x,t)}{t^{n/p+1}} dt \\ &\quad \le C \left\{ \int_r^\infty \frac{u^p(x,t)}{t^{n-\alpha/p+1}} dt \right\}^{1/p} \left\{ \int_r^\infty \frac{1}{t^{\alpha p'+1}} dt \right\}^{1/p'} \\ &\quad \le C \frac{u(x,r)}{r^{n/p}}. \end{aligned} \quad (17)$$

From this, it follows that if $p(x) \equiv p, 1 < p < \infty$, is a constant function, then condition (12) is weaker than condition (16). Thus, the class of the Morrey spaces introduced in Definition 4 is more wide than that satisfying condition (1.8) in [21]. More studies of common Morrey spaces can be seen in [22, 23] and so forth.

(3) If $u(x,r) = |B(x,r)|^{1/p(x)-1/q(x)}$, $p(x) \le q(x)$, then the space mentioned in Definition 4 is the Morrey space with variable exponent introduced in [24]. And when $1 < s < \kappa_{p'(x)}$, $1/p(x) - 1/q(x) < 1 - 1/s$, it is easy to see $u(x,r)$ satisfying condition (12). That is because it follows from $p(x) \in \mathscr{B}(\mathbb{R}^n)$, $1 < s < \kappa_{p'(x)}$, that (see [20])

$$\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C 2^{jn(1/s-1)}, \quad \forall x \in \mathbb{R}^n, \ r > 0, \ j \in \mathbb{N}. \quad (18)$$

For Littlewood-Paley operators $S_{\psi,a}$, g_ψ, and g_μ^*, in this paper, we have the following results.

Theorem 6. *Suppose that function $\psi \in L^1(\mathbb{R}^n)$ satisfies (i)–(iii) and $S_{\psi,a}$ is defined by* (1). *If $u \in \mathbb{W}_{p(\cdot)}$, $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $1 < r < \infty$, then there exists a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^\infty$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:*

$$\left\| \left\{ \sum_h \left| S_{\psi,a}(f_h) \right|^r \right\}^{1/r} \right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \quad (19)$$

Theorem 7. *Suppose that g_ψ is defined by* (3). *Then under the same condition as the one in Theorem 6, there exists a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^\infty$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:*

$$\left\| \left\{ \sum_h \left| g_\psi(f_h) \right|^r \right\}^{1/r} \right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \quad (20)$$

Theorem 8. *Suppose that g_μ^* is defined by* (4) *and $\mu > 3 + 2(\varepsilon + \gamma)/n$, $0 < \gamma < \varepsilon$. Then under the same condition as the one in Theorem 6, there exists a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^\infty$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:*

$$\left\| \left\{ \sum_h \left| g_\mu^*(f_h) \right|^r \right\}^{1/r} \right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \quad (21)$$

For commutators $[b^m, S_{\psi,a}]$, $[b^m, g_\psi]$, and $[b^m, g_\mu^*]$, we have the following results.

Theorem 9. *Suppose that function $\psi \in L^1(\mathbb{R}^n)$ satisfies (i)–(iii) and $[b^m, S_{\psi,a}]$ is defined by* (5). *Let $b \in BMO(\mathbb{R}^n)$, $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $m \ge 1$, $1 < r < \infty$. If, for any $x \in \mathbb{R}^n$ and $r_0 > 0$, function u satisfies*

$$\sum_{j=0}^\infty \frac{\|\chi_{B(x,r_0)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r_0)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} j^m u\left(x, 2^{j+1} r_0\right) \le C u(x, r_0), \quad (22)$$

then there exists a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^\infty$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:

$$\begin{aligned} &\left\| \left\{ \sum_h \left| \left[b^m, S_{\psi,a} \right](f_h) \right|^r \right\}^{1/r} \right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \\ &\quad \le C \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \end{aligned} \quad (23)$$

Theorem 10. *Suppose that $[b^m, g_\psi]$ is defined by* (6). *Then under the same condition as the one in Theorem 9, there exists*

a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^{\infty}$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:

$$\left\|\left\{\sum_h \left|\left[b^m, g_\psi\right](f_h)\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \tag{24}$$

Theorem 11. *Suppose that $[b^m, g_\mu^*]$ is defined by (7), $\mu > 3 + 2(\varepsilon + \gamma)/n$, and $0 < \gamma < \varepsilon$. Then under the same condition as the one in Theorem 9, there exists a constant $C > 0$ independent of f such that, for any function sequences $\{f_h\}_{h=1}^{\infty}$ with $\|\{\sum_h |f_h|^r\}^{1/r}\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} < \infty$, the following inequality holds:*

$$\left\|\left\{\sum_h \left|\left[b^m, g_\mu^*\right](f_h)\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \tag{25}$$

Remark 12. (1) It is easy to see that condition (22) in Theorem 9 is stronger than condition (12) in Definition 3. Therefore, if a function u satisfies condition (22), then $u \in \mathbb{W}_{p(\cdot)}$.

(2) The function u which satisfies (22) exists. In fact, if we take $\tau : 0 \le \tau < 1/e_{p(\cdot)}$ such that function u satisfies

$$\frac{u(x, 2r)}{u(x, r)} \le 2^{n\tau}, \quad \forall x \in \mathbb{R}^n,\ r > 0, \tag{26}$$

then $u \in \mathbb{W}_{p(\cdot)}$. That is because, for any $\tau < 1/e_{p(\cdot)}$, there exists $s < \kappa_{p'(\cdot)}$ such that $\tau < 1 - 1/s < 1 - 1/\kappa_{p'(\cdot)} = 1/e_{p(\cdot)}$. Hence, it follows from (18) that

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} j^m u\left(x, 2^{j+1}r\right) \le C\sum_{j=0}^{\infty} j^m 2^{-jn(1-1/s)} 2^{jn\tau} u(x, r) \le Cu(x, r). \tag{27}$$

We end this section by introducing some conventional notations which will be used later. Throughout this paper, given a function f, we denote the mean value of f on E by $f_E =: (1/|E|)\int_E f(x)dx$. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$; namely, $1/p(x) + 1/p'(x) = 1$ holds. C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2. Preliminary Lemmas

In this section, we introduce some conclusions which will be used in the proofs of our main results.

Lemma 13 (see [10] (generalized Hölder's inequality)). *Let $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathscr{P}(\mathbb{R}^n)$.*

(1) *For any $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $g \in L^{p'(\cdot)}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le C_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{28}$$

where $C_p = 1 + 1/p_- - 1/p_+$.

(2) *For any $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, when $1/p(x) = 1/p_1(x) + 1/p_2(x)$, one has*

$$\|f(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_{p_1,p_2} \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}, \tag{29}$$

where $C_{p_1,p_2} = (1 + 1/p_{1_-} - 1/p_{1_+})^{1/p_-}$.

Lemma 14 (see [17]). *If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then there exist constants $\delta_1, \delta_2, C > 0$, such that, for all balls $B \subset \mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\begin{aligned} \frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} &\le C\frac{|B|}{|S|}, \\ \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} &\le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \\ \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} &\le C\left(\frac{|S|}{|B|}\right)^{\delta_2}. \end{aligned} \tag{30}$$

Remark 15. From formula (12), it follows that

$$\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u\left(x, 2^{j+1}r\right) \le Cu(x, r), \quad \forall j \in \mathbb{N}. \tag{31}$$

Thus, by Lemma 14, we have, $\forall j \in \mathbb{N}$,

$$\frac{u\left(x, 2^j r\right)}{u(x, r)} \le C\frac{\|\chi_{B(x,2^j r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{\left|B\left(x, 2^j r\right)\right|}{|B(x, r)|} \le C2^{nj}. \tag{32}$$

Lemma 16 (see [18]). *If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then there exists constant $C > 0$, such that, for all balls $B \subset \mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C. \tag{33}$$

Lemma 17 (see [25]). *Let $b \in BMO(\mathbb{R}^n)$; m is a positive integer. There exist constants $C > 0$, such that, for any $k, j \in \mathbb{Z}$ with $k > j$,*

(1) $C^{-1}\|b\|_*^m \le \sup_B (1/\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}) \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*^m$;

(2) $\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(k - j)^m \|b\|_*^m \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 18 (see [26]). *Let $\psi \in L^1(\mathbb{R}^n)$ satisfy (i)–(iii). If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $1 < q < \infty$, then for all bounded compactly support functions f_j such that $\{f_j\}_{j=1}^{\infty} \in L^{p(\cdot)}(l^q)$, that is, $\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$, the following vector-valued inequalities hold:*

(1) $\|(\sum_j |S_{\psi,a}(f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(2) $\|(\sum_j |g_\psi(f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(3) $\|(\sum_j |g_\mu^*(f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

where $\mu > 2$, $0 < \gamma < \min\{(\mu-2)n/2, \varepsilon\}$.

Lemma 19 (see [26]). *Let* $b \in BMO, \psi \in L^1(\mathbb{R}^n)$ *satisfy (i)–(iii),* $m \in \mathbb{N} \setminus \{0\}$. *If* $p(\cdot) \in \mathscr{B}(\mathbb{R}^n), 1 < q < \infty$, *then for all bounded compactly support functions* f_j *such that* $\{f_j\}_{j=1}^\infty \in L^{p(\cdot)}(l^q)$, *that is,* $\|(\sum_j(|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$, *the following vector-valued inequalities hold:*

(1) $\|(\sum_j |[b^m, S_{\psi,a}](f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$
$\le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(2) $\|(\sum_j |[b^m, g_\psi](f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$
$\le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(3) $\|(\sum_j |[b^m, g_\mu^*](f_j)|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$
$\le C\|(\sum_j |f_j|^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

where $\mu > 2$, $0 < \gamma < \min\{(\mu-2)n/2, \varepsilon\}$.

3. Proofs of Main Results

Next, let us show the proofs of Theorems 6–11, respectively.

Proof of Theorem 6. Let $\|\{f_h\}_h\|_{l^r} \in \mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)$; for any $x_0 \in \mathbb{R}^n$, $r_0 > 0$, denote

$$f_h(x) \triangleq f_h^0(x) + \sum_{j=1}^\infty f_h^j(x), \tag{34}$$

where $f_h^0 = f_h\chi_{B(x_0,2r_0)}$, $f_h^j = f_h\chi_{B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)}$, $j \in \mathbb{N} \setminus \{0\}$.

Noting that, in order to prove Theorem 6, it is enough to show that the following inequality holds:

$$\begin{aligned}&\frac{1}{u(x_0,r_0)}\left\|\left\{\sum_h \left|S_{\psi,a}(f_h)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad\le C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}.\end{aligned} \tag{35}$$

Thus,

$$\begin{aligned}&\frac{1}{u(x_0,r_0)}\left\|\left\{\sum_h \left|S_{\psi,a}(f_h)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad\le \frac{C}{u(x_0,r_0)}\left\|\left\{\sum_h \left|S_{\psi,a}(f_h^0)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\qquad+\frac{C}{u(x_0,r_0)}\sum_{j=1}^\infty\left\|\left\{\sum_h \left|S_{\psi,a}(f_h^j)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad\triangleq D_1 + D_2.\end{aligned} \tag{36}$$

For the term D_1, notice that supp $f_h^0 \subset B(x_0, 2r_0)$; using Lemma 18 and (32), it is easy to see that

$$\begin{aligned}D_1 &\le \frac{C}{u(x_0,r_0)}\left\|\left\{\sum_h \left|f_h^0\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\frac{u(x_0,2r_0)}{u(x_0,r_0)}\frac{1}{u(x_0,2r_0)}\\ &\quad\times\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\chi_{B(x_0,2r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C2^n\sup_{y\in\mathbb{R}^n,R>0}\frac{1}{u(y,R)}\\ &\quad\times\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\chi_{B(y,R)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}.\end{aligned} \tag{37}$$

We now turn to estimate D_2. To do this, we need to consider $S_{\psi,a}$ first. Without loss of generality, we may assume that $a \ge 1$. Let

$$\begin{aligned}\Gamma' &= \left\{y \in \mathbb{R}^n : |y-x| < at,\ y \le 2^{j+1}r_0\right\},\\ \Gamma'' &= \left\{y \in \mathbb{R}^n : |y-x| < at,\ y > 2^{j+1}r_0\right\}.\end{aligned} \tag{38}$$

Then, by Minkowski's inequality, we have

$$\begin{aligned}
&S_{\psi,a}\left(f_h^j\right)(x)\\
&\quad=\left(\int_{\Gamma_a(x)}\left|\int_{\mathbb{R}^n}t^{-n}\psi\left(\frac{y-z}{t}\right)f_h^j(z)\,dz\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}\\
&\quad\le\int_{\mathbb{R}^n}\left|f_h^j(z)\right|\left(\int_{\Gamma_a(x)}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\right)^{1/2}dz\\
&\quad\le\int_{\mathbb{R}^n}\left|f_h^j(z)\right|\left(\int_0^\infty\int_{\Gamma'}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\right)^{1/2}dz\\
&\qquad+\int_{\mathbb{R}^n}\left|f_h^j(z)\right|\left(\int_0^\infty\int_{\Gamma''}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\right)^{1/2}dz.
\end{aligned}\tag{39}$$

Observe that if $x\in B(x_0,r_0)$, $z\in B(x_0,2^{j+1}r_0)\setminus B(x_0,2^jr_0)$, $j\ge1$, then

$$\begin{aligned}
t+|y-z|&\ge\frac{|x-y|}{a}+|y-z|\\
&\ge\frac{|x-y|+|y-z|}{a}\ge\frac{|z|-|x|}{a}\ge\frac{|z|}{2a}.
\end{aligned}\tag{40}$$

Therefore, it follows from condition (ii) that

$$\begin{aligned}
&\int_0^\infty\int_{\Gamma'}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\\
&\quad\le\int_0^\infty\int_{\Gamma'}t^{-3n-1}\left(1+\frac{|y-z|}{t}\right)^{-2(n+\varepsilon)}dy\,dt\\
&\quad\le\int_0^{2^{j+1}r_0}\int_{\Gamma'}\frac{t^{2\varepsilon-n-1}}{(t+|y-z|)^{2(n+\varepsilon)}}dy\,dt\\
&\qquad+\int_{2^{j+1}r_0}^\infty\int_{\Gamma'}\frac{t^{2\varepsilon-n-1}}{(t+|y-z|)^{2(n+\varepsilon)}}dy\,dt\\
&\quad\le C\frac{a^{2(n+\varepsilon)}}{|z|^{2(n+\varepsilon)}}\int_0^{2^{j+1}r_0}\int_{|x-y|<at}t^{2\varepsilon-n-1}dy\,dt\\
&\qquad+C\frac{a^{2n}}{|z|^{2n}}\int_{2^{j+1}r_0}^\infty\int_{|x-y|<at}t^{-3n-1}dy\,dt\\
&\quad\le C\frac{a^{2(n+\varepsilon)}}{|z|^{2(n+\varepsilon)}}\int_0^{2^{j+1}r_0}a^nt^{2\varepsilon-1}dy\,dt\\
&\qquad+C\int_{2^{j+1}r_0}^\infty a^nt^{-2n-1}dy\,dt\\
&\quad\le Ca^{3n+2\varepsilon}\left(2^jr_0\right)^{-2n}.
\end{aligned}\tag{41}$$

On the other hand, we denote

$$\begin{aligned}
&\int_0^\infty\int_{\Gamma''}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\\
&\quad\le C\int_0^\infty\int_{\Gamma''}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)-\psi\left(\frac{y}{t}\right)\right|^2dy\,dt\\
&\qquad+C\int_0^\infty\int_{\Gamma''}t^{-3n-1}\left|\psi\left(\frac{y}{t}\right)\right|^2dy\,dt\\
&\quad\triangleq I+II.
\end{aligned}\tag{42}$$

Note that if $y>2^{j+1}r_0$, then $t>|x-y|/a\ge(|x|-|y|)/a>(2^{j+1}r_0-r_0)/a\ge2^jr_0/a$. Thus, by condition (iii), we get

$$\begin{aligned}
I&\le C\int_{2^jr_0/a}^\infty\int_{\Gamma''}t^{-3n-1}\left(\frac{|z|}{t}\right)^{2\gamma}\left(1+\frac{|y|}{t}\right)^{-2(n+\gamma+\varepsilon)}dy\,dt\\
&\le C\left(2^jr_0\right)^{2\gamma}\int_{2^jr_0/a}^\infty\int_{|x-y|<at}t^{-3n-2\gamma-1}dy\,dt\\
&\le C\left(2^jr_0\right)^{2\gamma}a^n\int_{2^jr_0/a}^\infty t^{-2n-2\gamma-1}dy\,dt\\
&\le Ca^{3n+2\gamma}\left(2^jr_0\right)^{-2n}.
\end{aligned}\tag{43}$$

And by condition (ii), similar to the estimate of I, we obtain

$$\begin{aligned}
II&\le C\int_{2^jr_0/a}^\infty\int_{\Gamma''}t^{-3n-1}\left(\frac{|z|}{t}\right)^{2\gamma}\left(1+\frac{|y|}{t}\right)^{-2(n+\varepsilon)}dy\,dt\\
&\le C\int_{2^jr_0/a}^\infty\int_{\Gamma''}\frac{t^{2\varepsilon-n-1}}{(t+|y|)^{2(n+\varepsilon)}}dy\,dt\\
&\le Ca^{3n}\left(2^jr_0\right)^{-2n}.
\end{aligned}\tag{44}$$

Hence, from the estimates above, it follows that

$$S_{\psi,a}\left(f_h^j\right)(x)\le Ca^{3n/2+\gamma+\varepsilon}\left(2^jr_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}.\tag{45}$$

Thus,

$$\begin{aligned}
D_2&\le\frac{C}{u(x_0,r_0)}\sum_{j=1}^\infty\left\|\left\{\sum_h\left|a^{3n/2+\gamma+\varepsilon}\left(2^jr_0\right)^{-n}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le a^{3n/2+\gamma+\varepsilon}\frac{C}{u(x_0,r_0)}\sum_{j=1}^\infty\left(2^jr_0\right)^{-n}\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\qquad\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}.
\end{aligned}\tag{46}$$

Therefore, applying the generalized Hölder's inequality, Lemma 16, and (12), we have

$$\begin{aligned}
D_2 &\le a^{3n/2+\gamma+\varepsilon}\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^j r_0\right)^{-n}\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\le \frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^j r_0\right)^{-n}\left|B\left(x_0,2^{j+1}r_0\right)\right| \\
&\quad\times\frac{\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le \frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\frac{\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}u\left(x_0,2^{j+1}r_0\right) \\
&\quad\times\frac{1}{u(x_0,2^{j+1}r_0)}\left\|\left\{\sum_h\left|f_h\right|^r\right\}^{1/r}\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C\sup_{y\in\mathbb{R}^n,R>0}\frac{1}{u(y,R)}\left\|\left\{\sum_h\left|f_h\right|^r\right\}^{1/r}\chi_{B(y,R)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C\left\|\left\{\sum_h\left|f_h\right|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}.
\end{aligned}\tag{47}$$

Adding up the estimates of D_1, D_2, we obtain

$$\begin{aligned}
&\frac{1}{u(x_0,r_0)}\left\|\left\{\sum_h\left|S_{\psi,a}(f_h)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\le C\left\|\left\{\sum_h\left|f_h\right|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}.
\end{aligned}\tag{48}$$

This completes the proof of Theorem 6. □

Now let us prove Theorems 7 and 8 in brief.

Proof of Theorem 7. For g_ψ, similar to the estimate of $S_{\psi,a}$, via a simple calculation, we get that (see [26]) if $x \in B(x_0,r_0)$, supp $f_h^j \subset B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)$, $j \ge 1$, then

$$g_\psi\left(f_h^j\right)(x) \le C\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}. \tag{49}$$

Hence, similar to the proof of Theorem 6, it follows from inequality (2) in Lemma 18 that

$$\left\|\left\{\sum_h\left|g_\psi(f_h)\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h\left|f_h\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \tag{50}$$

This accomplishes the proof of Theorem 7. □

Proof of Theorem 8. For g_μ^*, by the definitions of $S_{\psi,a}$ and g_μ^*, we have

$$\begin{aligned}
&g_\mu^*(f)(x) \\
&\quad=\left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\mu n}\left|\psi_t * f(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2} \\
&\quad\le\left(\int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\mu n}\left|\psi_t * f(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2} \\
&\qquad+\sum_{l=1}^{\infty}\left(\int_0^\infty\int_{2^{l-1}t\le|x-y|<2^l t}\left(\frac{t}{t+|x-y|}\right)^{\mu n}\right. \\
&\qquad\qquad\left.\times\left|\psi_t * f(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2} \\
&\quad\le S_{\psi,a}\left(f_h^j\right)(x)+\sum_{l=1}^{\infty}\left(1+2^{l-1}\right)^{-\mu n/2}S_{\psi,2^l a}\left(f_h^j\right)(x).
\end{aligned}\tag{51}$$

According to the estimate of $S_{\psi,a}$ in the proof of Theorem 6, we know that if $x \in B(x_0,r_0)$, supp $f_h^j \subset B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)$, $j \ge 1$, then

$$S_{\psi,a}\left(f_h^j\right)(x) \le Ca^{3n/2+\varepsilon+\gamma}\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}. \tag{52}$$

Thus, as $\mu > 3+2(\varepsilon+\gamma)/n$, we obtain

$$\begin{aligned}
g_\mu^*\left(f_h^j\right)(x) &\le Ca^{3n/2+\varepsilon+\gamma}\left(1+\sum_{l=1}^{\infty}2^{(3n/2+\varepsilon+\gamma-\mu n/2)}\right) \\
&\quad\times\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)} \\
&\le Ca^{3n/2+\varepsilon+\gamma}\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}.
\end{aligned}\tag{53}$$

Hence, also similar to the proof of Theorem 6, and from inequality (3) in Lemma 18, it follows that

$$\left\|\left\{\sum_h\left|g_\mu^*(f_h)\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h\left|f_h\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \tag{54}$$

This finishes the proof of Theorem 8. □

Proof of Theorem 9. Let $b \in \mathrm{BMO}$, $\|\{f_h\}_h\|_{l^r} \in \mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)$. For any $x_0 \in \mathbb{R}^n$, $r_0 > 0$, denote

$$f_h(x) \triangleq f_h^0(x)+\sum_{j=1}^{\infty}f_h^j(x), \tag{55}$$

where $f_h^0 = f_h\chi_{B(x_0,2r_0)}$ and $f_h^j = f_h\chi_{B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)}$, $j \in \mathbb{N}\setminus\{0\}$.

To finish the proof of Theorem 9, we only need to prove

$$\frac{1}{u(x_0,r_0)}\left\|\left\{\sum_h\left|\left[b^m,S_{\psi,a}\right](f_h)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}. \tag{56}$$

Thus,

$$\begin{aligned}\frac{1}{u(x_0,r_0)}&\left\|\left\{\sum_h\left|\left[b^m,S_{\psi,a}\right](f_h)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le \frac{C}{u(x_0,r_0)}\left\|\left\{\sum_h\left|\left[b^m,S_{\psi,a}\right](f_h^0)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad+\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left\|\left\{\sum_h\left|\left[b^m,S_{\psi,a}\right](f_h^j)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\triangleq E_1+E_2.\end{aligned} \tag{57}$$

For the term E_1, notice that supp $f_h^0 \subset B(x_0,2r_0)$; by Lemma 19 and inequality (32), we have

$$\begin{aligned}E_1 &\le \frac{C}{u(x_0,r_0)}\left\|\left\{\sum_h\left|f_h^0\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\frac{u(x_0,2r_0)}{u(x_0,r_0)}\frac{1}{u(x_0,2r_0)}\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\chi_{B(x_0,2r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C2^n\sup_{y\in\mathbb{R}^n,R>0}\frac{1}{u(y,R)}\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\chi_{B(y,R)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}.\end{aligned} \tag{58}$$

Now we turn to estimate E_2. According to the estimate of $S_{\psi,a}$ in the proof of Theorem 6, we see that if $x\in B(x_0,r_0)$, $z\in B(x_0,2^{j+1}r_0)\setminus B(x_0,2^jr_0)$, $j\ge 1$, then

$$S_{\psi,a}\left(f_h^j\right)(x)\le Ca^{3n/2+\varepsilon+\gamma}\left(2^jr_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}. \tag{59}$$

Therefore,

$$\begin{aligned}&\left|\left[b^m,S_{\psi,a}\right]\left(f_h^j\right)(x)\right|\\ &\quad=\left|S_{\psi,a}\left[(b(x)-b)^mf\right](x)\right|\\ &\quad\le Ca^{3n/2+\varepsilon+\gamma}\left(2^jr_0\right)^{-n}\left\|(b(\cdot)-b)^mf_h^j\right\|_{L^1(\mathbb{R}^n)}.\end{aligned} \tag{60}$$

Thus,

$$\begin{aligned}E_2&\le\frac{C}{u(x_0,r_0)}\\ &\quad\times\sum_{j=1}^{\infty}\left\|a^{3n/2+\gamma+\varepsilon}\left(2^jr_0\right)^{-n}\left\{\sum_h\left\|(b(\cdot)-b)^mf_h^j\right\|_{L^1(\mathbb{R}^n)}^r\right\}^{1/r}\right.\\ &\qquad\left.\times\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^jr_0\right)^{-n}\\ &\quad\times\left\|\left\|\left\{\sum_h\left|(b(\cdot)-b)^mf_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^jr_0\right)^{-n}\left\|\left(b-b_{B(x_0,r_0)}\right)^m\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}\\ &\quad+\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^jr_0\right)^{-n}\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad\times\left\|\left\{\sum_h\left|\left(b-b_{B(x_0,r_0)}\right)^mf_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}.\end{aligned} \tag{61}$$

Using Hölder's inequality and Lemma 17, we get

$$\begin{aligned}E_2&\le\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^jr_0\right)^{-n}\|b\|_*^m\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ &\quad+\frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^jr_0\right)^{-n}\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad\times\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad\times\left\|\left(b-b_{B(x_0,r_0)}\right)^m\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

$$\leq \frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^j r_0\right)^{-n}\left[j^m+1\right]\|b\|_*^m \times \left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \times \left\|\left\{\sum_h \left|f_h^j\right|^r\right\}^{\frac{1}{r}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{62}$$

And then, it follows from Lemma 16 and (22) that

$$\begin{aligned} E_2 &\leq C\|b\|_*^m \frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty}\left(2^j r_0\right)^{-n} j^m \left|B\left(x_0,2^{j+1}r_0\right)\right| \\ &\quad\times \frac{\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \times u\left(x_0,2^{j+1}r_0\right)\frac{1}{u\left(x_0,2^{j+1}r_0\right)} \\ &\quad\times \left\|\left\{\sum_h \left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C\|b\|_*^m \frac{C}{u(x_0,r_0)}\sum_{j=1}^{\infty} j^m \frac{\left\|\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B(x_0,2^{j+1}r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\ &\quad\times u\left(x_0,2^{j+1}r_0\right) \\ &\quad\times \sup_{y\in\mathbb{R}^n,R>0}\frac{1}{u(y,R)}\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\chi_{B(y,R)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}. \end{aligned} \tag{63}$$

Hence, Adding up the results of E_1, E_2, we have

$$\begin{aligned} &\frac{1}{u(x_0,r_0)}\left\|\left\{\sum_h \left|\left[b^m,S_{\psi,a}\right](f_h)\right|^r\right\}^{1/r}\chi_{B(x_0,r_0)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\quad\leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{M_{p(\cdot),u}(\mathbb{R}^n)}. \end{aligned} \tag{64}$$

The proof of Theorem 9 is accomplished. □

Proof of Theorem 10. For $[b^m, g_\psi]$, according to the estimate of g_ψ in the proof of Theorem 7, we see that if $x \in B(x_0,r_0)$, supp $f_h^j \subset B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)$, $j \geq 1$, then

$$g_\psi\left(f_h^j\right)(x) \leq C\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}. \tag{65}$$

Thus,

$$\begin{aligned} \left[b^m,g_\psi\right] &= \left|g_\psi\left[(b(x)-b)^m f\right](x)\right| \\ &\leq C\left(2^j r_0\right)^{-n}\left\|(b(\cdot)-b)^m f_h^j\right\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{66}$$

Hence, similar to the proof of Theorem 9, and from inequality (2) in Lemma 19, it follows that

$$\begin{aligned} &\left\|\left\{\sum_h \left|g_\psi(f_h)\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \\ &\quad\leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \end{aligned} \tag{67}$$

The proof of Theorem 10 is completed. □

Proof of Theorem 11. For $[b^m, g_\mu^*]$, according to the estimate of $[b^m, g_\mu^*]$ in the proof of Theorem 8, we get that if $x \in B(x_0,r_0)$, supp $f_h^j \subset B(x_0,2^{j+1}r_0)\setminus B(x_0,2^j r_0)$, $j \geq 1$, then

$$g_\mu^*\left(f_h^j\right)(x) \leq Ca^{3n/2+\varepsilon+\gamma}\left(2^j r_0\right)^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}. \tag{68}$$

Thus,

$$\begin{aligned} \left[b^m,g_\mu^*\right] &= \left|g_\mu^*\left[(b(x)-b)^m f\right](x)\right| \\ &\leq Ca^{3n/2+\varepsilon+\gamma}\left(2^j r_0\right)^{-n}\left\|(b(\cdot)-b)^m f_h^j\right\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{69}$$

Hence, also similar to the proof of Theorem 9, it follows from inequality (3) in Lemma 19 that

$$\begin{aligned} &\left\|\left\{\sum_h \left|\left[b^m,g_\mu^*\right](f_h)\right|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)} \\ &\quad\leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{\mathscr{M}_{p(\cdot),u}(\mathbb{R}^n)}. \end{aligned} \tag{70}$$

The proof of Theorem 11 is accomplished. □

Conflict of Interests

The authors declare that they have no conflict of interests.

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