Hindawi Publishing Corporation
The Scientific World Journal
Volume 2014, Article ID 901726, 10 pages
http://dx.doi.org/10.1155/2014/901726

Research Article

On F-Algebras M^p $(1 < p < \infty)$ of Holomorphic Functions

Romeo Meštrović

Maritime Faculty, University of Montenegro, Dobrota 36, 85330 Kotor, Montenegro

Correspondence should be addressed to Romeo Meštrović; romeo@ac.me

Received 31 August 2013; Accepted 4 November 2013; Published 28 January 2014

Academic Editors: A. Ibeas and G.-Q. Xu

We consider the classes M^p $(1 < p < \infty)$ of holomorphic functions on the open unit disk $\mathbb{D}$ in the complex plane. These classes are in fact generalizations of the class M introduced by Kim (1986). The space M^p equipped with the topology given by the metric ρ_p defined by $\rho_p(f,g) = \|f-g\|_p = (\int_0^{2\pi}\log^p(1+M(f-g)(\theta))(d\theta/2\pi))^{1/p}$, with $f,g \in M^p$ and $Mf(\theta) = \sup_{0\leqslant r<1}|f(re^{i\theta})|$, becomes an F-space. By a result of Stoll (1977), the Privalov space N^p $(1 < p < \infty)$ with the topology given by the Stoll metric d_p is an F-algebra. By using these two facts, we prove that the spaces M^p and N^p coincide and have the same topological structure. Consequently, we describe a general form of continuous linear functionals on M^p (with respect to the metric ρ_p). Furthermore, we give a characterization of bounded subsets of the spaces M^p. Moreover, we give the examples of bounded subsets of M^p that are not relatively compact.

1. Introduction and Preliminaries

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let $\mathbb{T}$ denote the boundary of $\mathbb{D}$. Let $L^q(\mathbb{T})$ $(0 < q \leq \infty)$ be the familiar Lebesgue spaces on the unit circle $\mathbb{T}$.

Following Kim ([1, 2]), the class M consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\int_0^{2\pi} \log^+ Mf(\theta)\,\frac{d\theta}{2\pi} < \infty, \tag{1}$$

where $\log^+|a| = \max\{\log a, 0\}$ and

$$Mf(\theta) = \sup_{0\leqslant r<1}\left|f\left(re^{i\theta}\right)\right| \tag{2}$$

is the maximal radial function of f. The Privalov class N^p $(1 < p < \infty)$ consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\sup_{0<r<1}\int_0^{2\pi}\left(\log^+\left|f\left(re^{i\theta}\right)\right|\right)^p\frac{d\theta}{2\pi} < +\infty. \tag{3}$$

These classes were firstly considered by Privalov in [3, page 93], where N^p is denoted as A_q.

Notice that for $p = 1$, the condition (3) defines the Nevanlinna class N of holomorphic functions in $\mathbb{D}$. Recall that the Smirnov class N^+ is the set of all functions f holomorphic on $\mathbb{D}$ such that

$$\lim_{r\to 1}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|\frac{d\theta}{2\pi} = \int_0^{2\pi}\log^+\left|f^*\left(e^{i\theta}\right)\right|\frac{d\theta}{2\pi} < +\infty, \tag{4}$$

where f^* is the boundary function of f on $\mathbb{T}$; that is,

$$f^*\left(e^{i\theta}\right) = \lim_{r\to 1-} f\left(re^{i\theta}\right) \tag{5}$$

is the radial limit of f which exists for almost every $e^{i\theta}$. We denote by H^q $(0 < q \leq \infty)$ the classical Hardy space on $\mathbb{D}$. It is known (see [4, 5]) that

$$N^r \subset N^p \quad (r > p), \qquad \bigcup_{q>0} H^q \subset \bigcap_{p>1} N^p, \qquad \bigcup_{p>1} N^p \subset M \subset N^+ \subset N, \tag{6}$$

where the above containment relations are proper.

The study of the spaces N^p $(1 < p < \infty)$ was continued in 1977 by Stoll [6] (with the notation $(\log^+ H)^\alpha$ in [6]). Further, the topological and functional properties of these spaces were studied in [4, 5, 7–14]; typically, the notation varied and these spaces are called the Privalov spaces in [12–15].

It is well known [16, page 26] that a function $f \in N^+$ if and only if $f = IF$, where I is an inner function on $\mathbb{D}$ and F is an outer function given by

$$F(z) = \exp\left(\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log\left|F^*\left(e^{it}\right)\right| \frac{dt}{2\pi}\right), \tag{7}$$

where $\log|F^*| \in L^1(\mathbb{T})$.

Privalov [3, page 98] showed that $f \in N^p$ if and only if $f = IF$, where I is an inner function on $\mathbb{D}$ and F is an outer function as given above with $\log^+|f^*| \in L^p(\mathbb{T})$.

Stoll [6, Theorem 4.2] showed that the space N^p (with the notation $(\log^+ H)^\alpha$ in [6]) with the topology given by the metric d_p defined by

$$d_p(f,g) = \left(\int_0^{2\pi} \left(\log\left(1+\left|f^*\left(e^{i\theta}\right)-g^*\left(e^{i\theta}\right)\right|\right)\right)^p \frac{d\theta}{2\pi}\right)^{1/p}, \quad f,g \in N^p \tag{8}$$

becomes an F-algebra. Recall that the function $d_1 = d$ defined on the Smirnov class N^+ by (8) with $p = 1$ induces the metric topology on N^+. Yanagihara [17] showed that, under this topology, N^+ is an F-space.

Furthermore, in connection with the spaces N^p $(1 < p < \infty)$, Stoll [6] (also see [7] and [12, Section 3]) also studied the spaces F^q $(0 < q < \infty)$ (with the notation $F_{1/q}$ in [6]), consisting of those functions f holomorphic on $\mathbb{D}$ for which

$$\lim_{r\to 1}(1-r)^{1/q}\log^+ M_\infty(r,f) = 0, \tag{9}$$

where

$$M_\infty(r,f) = \max_{|z|\le r}|f(z)|. \tag{10}$$

Stoll [6, Theorem 3.2] proved that the space F^q with the topology given by the family of seminorms $\{\|\cdot\|_{q,c}\}_{c>0}$ defined for $f \in F^q$ as

$$\|\|f\|\|_{q,c} = \sum_{n=0}^{\infty}\left|\widehat{f}(n)\right| e^{-cn^{1/(q+1)}} < \infty, \tag{11}$$

for each $c > 0$, where $\widehat{f}(n)$ is the nth Taylor coefficient of f, becomes a countably normed Fréchet algebra. By a result of Eoff [7, Theorem 4.2], F^p is the Fréchet envelope of N^p, and hence F^p and N^p have the same topological duals.

Here, as always in the sequel, we will need some of Stoll's results concerning the spaces F^q only with $1 < q < \infty$, and hence we will assume that $q = p > 1$ is any fixed number.

The study of the class M has been extensively investigated by Kim in [1, 2], Gavrilov and Zaharyan [18], and Nawrocky [19]. Kim [2, Theorems 3.1 and 6.1] showed that the space M with the topology given by the metric ρ defined by

$$\rho(f,g) = \int_0^{2\pi} \log\left(1+M(f-g)(\theta)\right)\frac{d\theta}{2\pi}, \quad f,g \in M \tag{12}$$

becomes an F-algebra. Furthermore, Kim [2, Theorems 5.2 and 5.3] gave an incomplete characterization of multipliers of M into H^∞. Consequently, the topological dual of M is not exactly determined in [2], but, as an application, it was proved in [2, Theorem 5.4] (also cf. [19, Corollary 4]) that M is not locally convex space. Furthermore, the space M is not locally bounded ([2, Theorem 4.5] and [19, Corollary 5]).

Although the class M is essentially smaller than the class N^+, Nawrocky [19] showed that the class M and the Smirnov class N^+ have the same corresponding locally convex structure which was already established by Yanagihara for the Smirnov class in [17, 20]. More precisely, it was proved in [19, Theorem 1] that the Fréchet envelope of the class M can be identified with the space F^+ of holomorphic functions on the open unit disk $\mathbb{D}$ such that

$$\|\|f\|\|_c := \sum_{n=0}^{\infty}\left|\widehat{f}(n)\right| e^{-c\sqrt{n}} < \infty, \tag{13}$$

for each $c > 0$, where $\widehat{f}(n)$ is the nth Taylor coefficient of f. Notice that F^+ coincides with the space F^1 defined above. It was shown in [17, 21] that F^+ is actually the containing Fréchet space for N^+. Moreover, Nawrocky [19, Theorem 1] characterized the set of all continuous linear functionals on M which by a result of Yanagihara [17] coincides with those on the Smirnov class N^+.

Motivated by the mentioned investigations of the classes M and N^+, and the fact that the classes N^p $(1 < p < \infty)$ are generalizations of the Smirnov class N^+, in Section 2, we consider the classes M^p $(1 < p < \infty)$ as generalizations of the class M. Accordingly, the *class* M^p $(1 < p < \infty)$ consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\int_0^{2\pi} \left(\log^+ Mf(\theta)\right)^p \frac{d\theta}{2\pi} < \infty. \tag{14}$$

Obviously,

$$\bigcup_{p>1} M^p \subset M. \tag{15}$$

Following [2], by analogy with the space M, the space M^p is equipped with the topology induced by the metric ρ_p defined as

$$\begin{aligned}\rho_p(f,g) &= \|f-g\|_p \\ &= \left(\int_0^{2\pi} \log^p\left(1+M(f-g)(\theta)\right)\frac{d\theta}{2\pi}\right)^{1/p},\end{aligned} \tag{16}$$

with $f,g \in M^p$.

In Section 2, we give the integral limit criterion for a function f holomorphic on the disk $\mathbb{D}$ to belong to the class M^p (Lemma 3). Furthermore, we prove that the space M^p is closed under integration (Theorem 4).

In Section 3 we study and compare the uniform convergence on compact subsets of $\mathbb{D}$ and the convergences induced by the metrics ρ_p and d_p in the space M^p, respectively. It is proved (Theorem 11) that $M^p = N^p$ for each $p > 1$.

It is proved in Section 4 that the space of all polynomials on $\mathbb{C}$ is a dense subset of M^p (Theorem 13). Hence, M^p is a separable metric space. We show that the space M^p with the topology given by the metric ρ_p becomes an F-space (Theorem 15). As an application, we prove that the metric spaces (M^p, ρ_p) and (N^p, d_p) have the same topological structure (Theorem 16). Consequently, we obtain a characterization of continuous linear functionals on M^p (Theorem 17). Notice that Theorem 17 with $p = 1$ characterizes the set of all continuous linear functionals on the space M, which is in fact the Nawrocky result [19, Theorem 1] mentioned above.

In Section 5 we obtain a characterization of bounded subsets of the spaces $M^p (= N^p)$ (Theorem 19). It is also given another necessary condition for a subset of M^p (N^p) to be bounded (Theorem 22). Finally, we give the examples of bounded subsets of M^p that are not relatively compact (Theorem 24).

2. The Classes M^p $(1 < p < \infty)$

Recall that, for a fixed $1 < p < \infty$, the class M^p consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\int_0^{2\pi} (\log^+ Mf(\theta))^p \frac{d\theta}{2\pi} < \infty. \tag{17}$$

Combining the inequalities $\log(|a| + 1) \le \log^+|a| + \log 2$ and $(|b| + |c|)^p \le 2^{p-1}(|b|^p + |c|^p)$, we obtain $\log^p(|a| + 1) \le 2^{p-1}((\log^+|a|)^p + (\log 2)^p)$ $(a, b, c \in \mathbb{C})$. The last inequality implies the fact that the condition (17) is equivalent to

$$\|f\|_p := \left(\int_0^{2\pi} (\log(1 + Mf(\theta)))^p \frac{d\theta}{2\pi}\right)^{1/p} < \infty. \tag{18}$$

Lemma 1. *The function $\|\cdot\|_p$ defined on M^p by (18) satisfies the following conditions:*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p \quad \forall f, g \in M^p, \tag{19}$$

$$\|fg\|_p \le \|f\|_p + \|g\|_p \quad \forall f, g \in M^p. \tag{20}$$

Hence, M^p is an algebra with respect to the pointwise addition and multiplication of functions.

Proof. Combining the inequality

$$\begin{aligned}&\log(1 + M(f + g)(\theta)) \\ &\quad \le \log(1 + Mf(\theta)) + \log(1 + Mg(\theta)), \\ &\qquad f, g \in M^p\end{aligned} \tag{21}$$

with Minkowski's integral inequality (with the power p), we immediately obtain (19). Similarly, combining the inequality

$$\begin{aligned}&\log(1 + M(fg)(\theta)) \\ &\quad \le \log(1 + Mf(\theta)) + \log(1 + Mg(\theta)), \\ &\qquad f, g \in M^p\end{aligned} \tag{22}$$

with Minkowski's integral inequality (with the exponent p), we obtain (20). □

Theorem 2. *The function ρ_p defined on M^p as*

$$\begin{aligned}\rho_p(f, g) &= \|f - g\|_p \\ &= \left(\int_0^{2\pi} \log^p(1 + M(f - g)(\theta)) \frac{d\theta}{2\pi}\right)^{1/p}, \\ &\qquad f, g \in M^p\end{aligned} \tag{23}$$

is a translation invariant metric on M^p. Further, the space M^p is a complete metric space with respect to the metric ρ_p.

Proof. If we suppose that $\rho_p(f, g) = 0$, for some $f, g \in M^p$, then by (23) it follows that $M(f - g)(\theta) = 0$ for almost every $\theta \in [0, 2\pi]$. Hence, $f^*(e^{i\theta}) = g^*(e^{i\theta})$ for almost every $e^{i\theta} \in \mathbb{T}$, and, by Riesz uniqueness theorem, we infer that $f(z) = g(z)$ for all $z \in \mathbb{D}$. As, by (19), the triangle inequality is satisfied, it follows that ρ_p is a metric on M^p. Finally, by the obvious inequality

$$\rho_p(f + h, g + h) = \rho_p(f, g), \quad f, g, h \in M^p, \tag{24}$$

we see that ρ_p is a translation invariant metric. This concludes the proof. □

For simplicity, here as always in the sequel, we shall write M^p instead of the metric space (M^p, ρ_p). For a function f holomorphic in $\mathbb{D}$ and for any fixed $0 \le \rho < 1$, denote by f_ρ the function defined on $\mathbb{D}$ as $f_\rho(z) = f(\rho z)$, $z \in \mathbb{D}$. Furthermore, for a given holomorphic function f on $\mathbb{D}$, let

$$Mf_\rho(\theta) = \sup_{0 \le r \le \rho} |f_r(\theta)| = \sup_{0 \le r \le \rho} \left|f\left(re^{i\theta}\right)\right|, \quad 0 \le \rho < 1. \tag{25}$$

Lemma 3. *A function f holomorphic on the unit disk $\mathbb{D}$ belongs to the class M^p if and only if it satisfies*

$$\begin{aligned}&\lim_{\rho \to 1} \int_0^{2\pi} \left(\log^+ Mf_\rho(\theta)\right)^p \frac{d\theta}{2\pi} \\ &\quad = \int_0^{2\pi} (\log^+ Mf(\theta))^p \frac{d\theta}{2\pi} < \infty.\end{aligned} \tag{26}$$

Proof. The condition (26) implies that $f \in M^p$. Conversely, assume that $f \in M^p$. Then

$$\begin{gathered}Mf_\rho(\theta) \longrightarrow Mf(\theta) \text{ as } \rho \longrightarrow 1 \\ \text{for almost every } \theta \in [0, 2\pi].\end{gathered} \tag{27}$$

Since, by the assumption, $f \in M^p$; that is, $\int_0^{2\pi} (\log^+ Mf(\theta))^p (d\theta/2\pi) < \infty$, using (27) and applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\rho \to 1} \int_0^{2\pi} \left(\log^+ Mf_\rho(\theta)\right)^p \frac{d\theta}{2\pi} = \int_0^{2\pi} (\log^+ Mf(\theta))^p \frac{d\theta}{2\pi}, \tag{28}$$

which completes the proof. □

Theorem 4. *The space M^p is closed under integration.*

Proof. For a given function $f \in M^p$, define

$$F(z) = \int_0^z f(z)\,dz = \int_0^r f(te^{i\theta}) e^{i\theta} dt. \quad (29)$$

It follows that $|F(re^{i\theta})| \le Mf(\theta)$, and thus $MF(\theta) \le Mf(\theta)$ for almost every $\theta \in [0, 2\pi]$. Therefore $F \in M^p$, as desired. □

3. Convergences in the Space M^p

Theorem 5. *For each function $f \in M^p$, $f_\rho \to f$ in M^p as $\rho \to 1-$.*

Proof. Assume that $f \in M^p$. Since $f \in N^+$, by Fatou's theorem, the radial limit $f^*(e^{i\theta}) = \lim_{r\to 1-} f(re^{i\theta})$ exists for almost every $\theta \in [0, 2\pi]$. Hence, for such a fixed θ, the function $t \mapsto f(te^{i\theta})$ is a continuous on $[0, 1]$, and thus it is uniformly continuous on $[0, 1]$. Therefore, for such a θ, we have

$$M(f - f_\rho)(\theta) \longrightarrow 0 \quad \text{as } \rho \longrightarrow 1-. \quad (30)$$

By the inequality

$$\begin{aligned} &\log\left(1 + M(f - f_\rho)(\theta)\right) \\ &\quad \le \log(1 + Mf(\theta)) + \log\left(1 + Mf_\rho(\theta)\right) \\ &\quad \le 2\log(1 + Mf(\theta)), \end{aligned} \quad (31)$$

in view of the fact that (18) is satisfied for $f \in M^p$, we obtain

$$\log^p\left(1 + M(f - f_\rho)(\theta)\right) \le 2^p \log^p(1 + Mf(\theta)) \in L^1(\mathbb{T}). \quad (32)$$

From this and (30), by the Lebesgue dominated convergence theorem, we obtain

$$\int_0^{2\pi} \left(\log\left(1 + M(f - f_\rho)(\theta)\right)\right)^p \frac{d\theta}{2\pi} \longrightarrow 0, \quad \text{as } \rho \longrightarrow 1-. \quad (33)$$

That is, $f_\rho \to f$ in M^p as $\rho \to 1-$. □

For the proof of completeness of the metric space (M^p, ρ_p) we will need the following lemmas.

Lemma 6. *If $\{f_n\}$ is a Cauchy sequence in M^p, then $(f_n)_\rho \to f_n$ in M^p as $\rho \to 1-$, where this convergence is uniform with respect to $n \in \mathbb{N}$.*

Proof. Suppose that $\{f_n\}$ is an arbitrary Cauchy sequence in M^p. Then for a given $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that

$$\rho_p(f_n, f_m) < \frac{\varepsilon}{3} \quad \forall n, m \ge k. \quad (34)$$

So by the triangle inequality, for each $n \ge k$, we have

$$\begin{aligned} \rho_p\left(f_n, (f_n)_\rho\right) &\le \rho_p(f_n, f_k) + \rho_p\left(f_k, (f_k)_\rho\right) \\ &\quad + \rho_p\left((f_k)_\rho, (f_n)_\rho\right) \\ &\le 2\rho_p(f_n, f_k) + \rho_p\left(f_k, (f_k)_\rho\right) \\ &< \frac{2\varepsilon}{3} + \rho_p\left(f_k, (f_k)_\rho\right). \end{aligned} \quad (35)$$

By Theorem 5, there exists $0 < \rho_0 < 1$ sufficiently near to 1, for which

$$\rho_p\left(f_l, (f_l)_\rho\right) < \frac{\varepsilon}{3} \quad \text{for each } \rho_0 < \rho < 1, \quad \text{for each } l = 1, \ldots, k. \quad (36)$$

Hence, by (35), we immediately obtain

$$\rho_p\left(f_n, (f_n)_\rho\right) < \varepsilon \quad \text{for each } \rho_0 < \rho < 1, \text{ for each } n \in \mathbb{N}. \quad (37)$$

This completes proof of Lemma 6. □

Lemma 7. *For any $p > 1$, $M^p \subseteq N^p$ and*

$$d_p(f, g) \le \rho_p(f, g) \quad \text{for each } f, g \in M^p, \quad (38)$$

where d_p is the metric of N^p defined by (8).

Proof. The inclusion $M^p \subseteq N^p$ is obvious, and (38) follows by the definition of the metrics d_p and ρ_p. □

Lemma 8. *The convergence with respect to the metric d_p of the space N^p is stronger than the metric of uniform convergence on compact subsets of the disk $\mathbb{D}$.*

Proof. The assertion immediately follows from the inequality on [5, page 898], which implies that, for any function $f \in N^p$ and $0 \le r < 1$, we have

$$\max_{|z|=r} |f(z)| \le \exp\left(\left(\frac{1+r}{1-r}\right)^{1/p} d_p(f, 0)\right). \quad (39)$$

□

Lemma 9. *If $\{f_n\}$ is a Cauchy sequence in the space M^p, then $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{D}$ to some holomorphic function f on $\mathbb{D}$.*

Proof. From the inequality (38) of Lemma 7, it follows that $\{f_n\}$ is a Cauchy sequence in N^p. Therefore, there exists $f \in N^p$ such that $f_n \to f$ in N^p, and so, by Lemma 8, $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. □

The following result is a maximal theorem of Hardy and Littlewood.

Lemma 10 (see [16, page 11]). *Let $1 < p \le +\infty$ and let φ be a function in the Lebesgue space $L^p(\mathbb{T})$. Let*

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2}\varphi(t)\,dt, \quad 0 \le r < 1 \tag{40}$$

be the Poisson integral of the function φ. Define

$$U(\theta) = \sup_{0\le r<1} |u(r,\theta)|, \quad \theta \in [0, 2\pi]. \tag{41}$$

Then $U \in L^p(\mathbb{T})$ and there is a constant A_p depending only on p such that

$$\|U\|_{L^p} \le A_p \|\varphi\|_{L^p}, \tag{42}$$

where $\|\cdot\|_{L^p}$ is the usual norm of the space $L^p(\mathbb{T})$.

We are now ready to state the following result.

Theorem 11. *$M^p = N^p$ for each $p > 1$; that is, the spaces M^p and N^p coincide.*

Proof. By Lemma 7, $M^p \subseteq N^p$ for each $p > 1$. For the proof of the converse of this inclusion, assume that $f \in N^p$. We will show that $f \in M^p$. As noticed in Section 1, f can be factorized as

$$f(z) = I(z)F(z), \quad z \in \mathbb{D}, \tag{43}$$

where $I(z)$ is the inner function and $F(z)$ is an outer function for the class N^p; that is,

$$F(z) = \omega \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log\left|f^*\left(e^{it}\right)\right|dt\right), \tag{44}$$

where ω is a constant of unit modulus. Furthermore, $\log^+|f^*| \in L^p(\mathbb{T})$. As $|I(z)| \le 1$, for each $z \in \mathbb{D}$, the previous factorization and the fact that $F \in M^p$ immediately imply that $f \in M^p$. Since

$$\Re\frac{e^{it}+z}{e^{it}-z} = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}, \quad z = re^{i\theta}, \tag{45}$$

from (44), we immediately obtain

$$\begin{aligned}\log\left|F\left(re^{i\theta}\right)\right| &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\log\left|f^*\left(e^{it}\right)\right|dt, \\ &\qquad 0 \le r < 1,\end{aligned} \tag{46}$$

whence it follows that, for $0 \le r < 1$,

$$\begin{aligned}\log^+\left|F\left(re^{i\theta}\right)\right| &= \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\log\left|f^*\left(e^{it}\right)\right|dt\right)^+ \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\log^+\left|f^*\left(e^{it}\right)\right|dt.\end{aligned} \tag{47}$$

The above inequality yields

$$\begin{aligned}\log^+ MF(\theta) &\le \sup_{0\le r<1}\left(\log^+\left|F\left(re^{i\theta}\right)\right|\right) \\ &\le \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\right. \\ &\qquad \left.\times \log^+\left|f^*\left(e^{it}\right)\right|dt\right).\end{aligned} \tag{48}$$

From the above inequality and the fact that $\log^+|f^*| \in L^p(\mathbb{T})$, we conclude by Lemma 10 that $\log^+ MF(\theta) \in L^p(\mathbb{T})$. This means that $F \in M^p$ and therefore $f \in M^p$. Thus $N^p \subseteq M^p$, and therefore $M^p = N^p$. This completes the proof. □

Corollary 12. *Let $f \in M^p$. Then*

$$\begin{aligned}&\int_0^{2\pi} \left(\log^+ Mf(\theta)\right)^p d\theta \\ &\quad \le C_p \int_0^{2\pi}\left(\log^+\left|f^*\left(e^{i\theta}\right)\right|\right)^p d\theta,\end{aligned} \tag{49}$$

where C_p is a nonnegative constant depending only on p.

Proof. Let F be the outer factor in the canonical factorization of $f \in M^p$. From the proof of Theorem 11, we see that for the functions $U(\theta) = \log^+ MF(\theta)$ and $\varphi(\theta) = \log^+|f^*(e^{i\theta})|$ the inequality (42) can be applied from Lemma 10. The obtained inequality is in fact (49) with F instead of f. Since $|f(z)| \le |F(z)|$, for each $z \in \mathbb{D}$, it follows that $Mf(\theta) \le MF(\theta)$ at almost every $\theta \in [0, 2\pi]$; thus (49) is obviously satisfied. □

4. M^p as an F-Algebra

Theorem 13. *The space of all polynomials over $\mathbb{C}$ is a dense subset of M^p. Hence, M^p is a separable metric space.*

Proof. Suppose that $f \in M^p$. Since, for a fixed $0 \le \rho < 1$, f_ρ is a holomorphic function on the closed unit disk $\overline{\mathbb{D}} : |z| \le 1$, by Runge's theorem, f_ρ can be uniformly approximated by polynomials on $\overline{\mathbb{D}}$. This together with the fact that, by Theorem 5, $f_\rho \to f$ in M^p as $\rho \to 1-$ yields that the space of all polynomials over $\mathbb{C}$ is a dense subset of M^p. Therefore, the set of all polynomials whose coefficients have rational real parts and rational imaginary parts becomes a countable dense subset of M^p. This concludes the proof. □

Theorem 14. *M^p is a complete metric space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in M^p. Then since N^p is complete, there is a $f \in N^p$ such that $f_n \to f$ in N^p. Since, by Theorem 11, $M^p = N^p$, it follows that $f \in M^p$, and thus it remains to show that $f_n \to f$ in M^p. By Theorem 5 and Lemma 6, there exist $0 < r < 1$ and $n_1 \in \mathbb{N}$ such that

$$\begin{aligned}&\rho_p\left(f_r, f\right) < \frac{\varepsilon}{3}, \\ &\rho_p\left(f_n, (f_n)_r\right) < \frac{\varepsilon}{3} \quad \text{for each } n \ge n_1.\end{aligned} \tag{50}$$

Since, by Lemma 9, a sequence $\{f_n\}$ converges uniformly on each closed disk $|z| \le \rho < 1$ to some function f, it follows that there exists $n_2 \in \mathbb{N}$ such that

$$\rho_p\left((f_n)_r, f_r\right) < \frac{\varepsilon}{3} \quad \text{for each } n \ge n_2. \tag{51}$$

Taking $n_0 = \max\{n_1, n_2\}$, by (50) and (51), the triangle inequality implies that

$$\rho_p\left(f_n, f\right) < \varepsilon \quad \forall n \ge n_0. \tag{52}$$

This shows that $f_n \to f$ in M^p, which completes the proof. □

Theorem 15. *M^p with the topology given by the metric ρ_p defined by (23) becomes an F-space.*

Proof. By [22, page 51], it suffices to show the following properties:

(i) ρ_p is an additive-invariant metric,

(ii) for any fixed $f \in M^p$, $c \mapsto cf$ is a continuous map from $\mathbb{C}$ into M^p,

(iii) for any fixed $c \in \mathbb{C}$, $f \mapsto cf$ is a continuous map from M^p into M^p, and

(iv) M^p is a complete metric space.

The assertion (i) follows from Theorem 2.

By the Lebesgue dominated convergence theorem, we have

$$\rho_p\left(cf, 0\right) = \left(\int_0^{2\pi} \log^p\left(1 + |c|\, Mf\left(\theta\right)\right) \frac{d\theta}{2\pi}\right)^{1/p} \longrightarrow 0 \quad \text{as } c \longrightarrow 0. \tag{53}$$

Let $k \in \mathbb{N}$ such that $|c| \le k$. Then the triangle inequality yields

$$\rho_p\left(cf, 0\right) \le \rho_p\left(kf, 0\right) \le k\rho_p\left(f, 0\right), \tag{54}$$

whence we see that $f \mapsto cf$ is a continuous map from M^p into M^p.

The assertion (iv) is in fact the assertion of Theorem 14.

This concludes the proof. □

We are now ready to prove that the (metric) spaces (M^p, ρ_p) and (N^p, d_p) have the same topological structure.

Theorem 16. *For each $p > 1$, the classes M^p and N^p coincide, and the metric spaces (M^p, ρ_p) and (N^p, d_p) have the same topological structure.*

Proof. Consider the identity map $j : M^p \to N^p$. Then, by the inequality (38) of Lemma 7, j is continuous. Since, by Theorem 11, $M^p = N^p$, j maps M^p onto N^p. Since M^p and N^p are both F-spaces, it follows, by the open mapping theorem [23, Corollary 2.12 (b)], that the inverse map j^{-1} of j is continuous. Hence, j is a homeomorphism, and so the metrics d_p and ρ_p induce the same topology on N^p and M^p, respectively. □

As an application of Theorem 16 and using the characterization of topological dual of the space F^p (which is by [7, Theorem 4.2] the Fréchet envelope of N^p) given by Stoll [6, Theorem 3.3] (cf. also [12, Theorem 3.5] and [13, Theorem 2]), we immediately get the following result.

Theorem 17. *If γ is a continuous linear functional on M^p, then there exists a sequence $\{\gamma_n\}_n$ of complex numbers with $\gamma_n = O(\exp(-cn^{1/(p+1)}))$, for some $c > 0$, such that*

$$\gamma\left(f\right) = \sum_{n=0}^{\infty} a_n \gamma_n, \tag{55}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in M^p$, with convergence being absolute. Conversely, if $\{\gamma_n\}$ is a sequence of complex numbers for which

$$\gamma_n = O\left(\exp\left(-cn^{1/(p+1)}\right)\right), \tag{56}$$

then (55) defines a continuous linear functional on M^p.

Corollary 18. *M^p is an F-algebra.*

Proof. By Theorem 15, M^p becomes an F-space. As N^p is an F-algebra, by Theorem 16, the multiplication is also continuous on M^p. Hence, M^p is an F-algebra. □

5. Bounded Subsets of M^p

It is proved in Section 4 (Theorem 16) that the spaces M^p and N^p coincide and have the same topological structure. Since N^p and M^p are not Banach spaces, it is of interest to obtain a characterization of bounded subsets of these spaces in terms of both metrics d_p and ρ_p.

Recall that, for a function $f \in N^p$, its boundary function f^* is defined as the radial limit $f^*(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta})$ which exists for almost every $e^{i\theta} \in \mathbb{T}$.

The following result gives a characterization of bounded subsets of $N^p (= M^p)$. Recall that the assertion (i)⇔(iii) is analogous to Theorem 1 in [21] that describes bounded subsets of N^+.

Theorem 19. *For given set $L \subset M^p$, the following conditions are equivalent:*

(i) *L is a bounded subset of M^p;*

(ii) *for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_E \left(\log^+ Mf\left(\theta\right)\right)^p \frac{d\theta}{2\pi} < \varepsilon \quad \forall f \in L, \tag{57}$$

for every measurable set $E \subset \mathbb{T}$ with the Lebesgue measure $|E| < \delta$;

(iii) *for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_E \left(\log^+ \left|f^*\left(e^{i\theta}\right)\right|\right)^p \frac{d\theta}{2\pi} < \varepsilon \quad \forall f \in L, \tag{58}$$

for each measurable set $E \subset \mathbb{T}$ with the Lebesgue measure $|E| < \delta$.

Proof. (ii)⇒(iii). It follows that from the obvious inequality $|f^*(e^{i\theta})| \le Mf(\theta)$, $f \in M^p$, for almost every $\theta \in [0, 2\pi]$.

(iii)⇒(i). Let

$$V = \{g \in N^p : d_p(g, 0) < \eta\} \quad (59)$$

be an arbitrary neighborhood of zero in N^p. Choose sufficiently small $\varepsilon > 0$ such that

$$\log^p(1+\varepsilon) + 2^{p-1}\log^p 2\delta + 2^{p-1}\varepsilon < \eta^p. \quad (60)$$

Now it follows that there exists δ, $0 < \delta < \varepsilon$, such that (iii) holds. Choose an $n \in \mathbb{N}$ for which $1/n < \delta$. Set

$$E_k = \left\{e^{i\theta} : \theta \in \left[\frac{2(k-1)\pi}{n}, \frac{2k\pi}{n}\right)\right\}, \quad k = 1, 2, \ldots, n. \quad (61)$$

Then $|E_k| = 1/n < \delta$, and thus by (iii) we have

$$\int_0^{2\pi} \left(\log^+\left|f^*\left(e^{i\theta}\right)\right|\right)^p \frac{d\theta}{2\pi} = \sum_{k=1}^{n} \int_{E_k} < n\varepsilon \quad \forall f \in L. \quad (62)$$

By (62) and Chebyshev's inequality, we conclude that for every function $f \in N^p$ there exists a measurable set $E_f \subset \mathbb{T}$ depending on f such that

$$\left|\mathbb{T} \setminus E_f\right| < \delta, \qquad \left(\log^+\left|f^*\left(e^{i\theta}\right)\right|\right)^p \le \frac{n\varepsilon}{\delta} \quad \text{on } E_f. \quad (63)$$

From (63), we obtain

$$\left|f^*\left(e^{i\theta}\right)\right| \le \exp\left(\frac{n\varepsilon}{\delta}\right)^{1/p} = K(\delta) = K \quad \text{on } E_f. \quad (64)$$

Choose α such that $0 < \alpha < \varepsilon/\delta$. Then using the inequality

$$\log^p(1+|a|) \le 2^{p-1}\left(\left(\log^+|a|\right)^p + \log^p 2\right), \quad (65)$$

(60) and (iii), for every $f \in L$, we obtain

$$\begin{aligned}
\left(d_p(\alpha f, 0)\right)^p &= \int_0^{2\pi} \log^p\left(1 + \left|\alpha f^*\left(e^{i\theta}\right)\right|\right)\frac{d\theta}{2\pi} \\
&= \int_{E_f} + \int_{\mathbb{T}\setminus E_f} \\
&\le \int_{E_f} \log^p(1+\varepsilon)\frac{d\theta}{2\pi} \\
&\quad + 2^{p-1}\left(\int_{\mathbb{T}\setminus E_f}\log^p 2\frac{d\theta}{2\pi} + \int_{\mathbb{T}\setminus E_f}\left(\log^+\left|f^*\left(e^{i\theta}\right)\right|\right)^p\frac{d\theta}{2\pi}\right) \\
&\le \log^p(1+\varepsilon) + 2^{p-1}\log^p 2\delta + 2^{p-1}\varepsilon \\
&< \eta^p.
\end{aligned} \quad (66)$$

Therefore, $d_p(\alpha f, 0) < \eta$, from which it follows that $\alpha L \subset V$. Hence, L is a bounded subset of N^p.

(i)⇒(ii). Assume that L is a bounded subset of M^p. Then for any given $\eta > 0$ there is a $\alpha_0 = \alpha_0(\eta)$, $0 < \alpha_0 < 1$, such that

$$\left(\rho_p(\alpha f, 0)\right)^p = \int_0^{2\pi} \log^p(1 + |\alpha| Mf(\theta))\frac{d\theta}{2\pi} < \eta^p \quad (67)$$

for each $f \in L$ and $|\alpha| \le \alpha_0$. It follows that

$$\int_0^{2\pi} \left(\log^+|\alpha| Mf(\theta)\right)^p \frac{d\theta}{2\pi} < \eta^p \quad \text{for each } f \in L, |\alpha| \le \alpha_0. \quad (68)$$

Since

$$\log^+ Mf(\theta) \le \log^+\alpha_0 Mf(\theta) + \log\frac{1}{\alpha_0}, \quad (69)$$

we obtain

$$\left(\log^+ Mf(\theta)\right)^p \le 2^{p-1}\left(\left(\log^+\alpha_0 Mf(\theta)\right)^p + \left(\log\frac{1}{\alpha_0}\right)^p\right). \quad (70)$$

For given $\varepsilon > 0$, choose $\eta > 0$ satisfying

$$\eta < \frac{\varepsilon^{1/p}}{2}, \quad (71)$$

and $\alpha_0 = \alpha_0(\eta)$ satisfying (67) and so also satisfying (68). Next, take $\delta > 0$ such that

$$\delta\log^p\frac{1}{\alpha_0} < \frac{\varepsilon}{2^p}. \quad (72)$$

Then for each set $E \subset \mathbb{T}$ with $|E| < \delta$, by (68)–(72), for every $f \in L$, we obtain

$$\begin{aligned}
\int_E \left(\log^+ Mf(\theta)\right)^p \frac{d\theta}{2\pi} &\le 2^{p-1}\left(\int_E\left(\log^+\alpha_0 Mf(\theta)\right)^p\frac{d\theta}{2\pi} + \int_E \log^p\frac{1}{\alpha_0}\frac{d\theta}{2\pi}\right) \\
&\le 2^{p-1}\eta^p + 2^{p-1}|E|\log^p\frac{1}{\alpha_0} \\
&\le \varepsilon.
\end{aligned} \quad (73)$$

Therefore, the condition (ii) of the theorem is satisfied, which concludes the proof. □

Remark 20. Note that the condition (ii) from Theorem 19 in fact means that the family $\{(\log^+ Mf(\theta))^p : f \in L\}$ is uniformly integrable on $\mathbb{T}$. The same assertion is also valid for the condition (iii). On the other hand, from the proof of Theorem 19, we see that (ii) implies that the family $\{(\log^+ Mf(\theta))^p : f \in L\}$ forms a bounded subset of the space $L^1(\mathbb{T})$; that is, there holds

$$\limsup_{f\in L}\int_0^{2\pi}\left(\log^+ Mf(\theta)\right)^p\frac{d\theta}{2\pi} < +\infty. \quad (74)$$

Similarly, it follows from (iii) that the family $\{(\log^+|f^*(e^{i\theta})|)^p : f \in L\}$ is bounded in $L^1(\mathbb{T})$.

Corollary 21. *If L is a subset of M^p for which the family*

$$\left\{\left(\log^+\left|f^*\left(e^{i\theta}\right)\right|\right)^p : f \in L\right\} \tag{75}$$

is uniformly integrable, then the family

$$\left\{\left(\log^+\left|f\left(re^{i\theta}\right)\right|\right)^p : f \in L, 0 \le r < 1\right\} \tag{76}$$

is also uniformly integrable.

Proof. The condition of Corollary 21 and (iii)⇒(ii) of Theorem 19 immediately yield that the family $\{(\log^+ Mf(\theta))^p : f \in L\}$ is uniformly integrable on the circle $\mathbb{T}$. This fact and the obvious inequality $|f(re^{i\theta})| \le Mf(\theta)$, $f \in M^p$, $0 \le r < 1$, for almost every $\theta \in [0, 2\pi]$, imply that the family $\{(\log^+|f(re^{i\theta})|)^p : f \in L, 0 \le r < 1\}$ is uniformly integrable. □

The following result gives a necessary condition for a subset of $M^p(= N^p)$ to be bounded.

Theorem 22. *Let L be a subset of M^p. If L is bounded in M^p, then*

$$M_\infty(r, f) \le K \exp\left(\frac{\omega(r)}{(1-r)^{1/p}}\right) \quad \text{for each } f \in L, \tag{77}$$

where $M_\infty(r, f) = \max_{0\le\theta<2\pi}|f(re^{i\theta})|$, K is a positive constant, and $\omega(r)$, $0 \le r < 1$, is a positive continuous function that does not depend on $f \in L$ and for which $\omega(r) \downarrow 0$ as $r \to 1$.

Proof. By the inequqlity (5.4) from the proof of Theorem 5.2 in [4], for all $f \in N^p$, we have

$$\left(\log^+\left|f\left(re^{i\theta}\right)\right|\right)^p \le \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\left(\log^+\left|f^*\left(e^{it}\right)\right|\right)^p dt. \tag{78}$$

As, by the assumption, L is a bounded subset of N^p, by Theorem 19 (iii), for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that

$$\int_0^{2\pi}\left(\log^+\left|f^*\left(e^{i\theta}\right)\right|\right)^p\frac{d\theta}{2\pi} < \frac{\varepsilon}{2} \quad \forall f \in L \tag{79}$$

and for every measurable set $E \subset \mathbb{T}$ with the Lebesgue measure $|E| < \delta$.

Further, from the proof of (iii)⇒(i) of Theorem 19, we see that for each $f \in N^p$ there is a measurable set $E_f \subset \mathbb{T}$ depending on f for which

$$\left|\mathbb{T}\setminus E_f\right| < \delta, \qquad \left(\log^+\left|f^*\left(e^{i\theta}\right)\right|\right)^p \le \frac{n\varepsilon}{\delta} \tag{80}$$

for almost every $e^{i\theta} \in E_f$. From (78)–(80), we obtain

$$\begin{aligned}\left(\log^+\left|f\left(re^{i\theta}\right)\right|\right)^p &= \int_{E_f} + \int_{E_f}\\ &\le \frac{n\varepsilon}{\delta} + \frac{1}{1-r}\frac{\varepsilon}{2},\end{aligned} \tag{81}$$

whence it follows that

$$(1-r)\left(\log^+ M_\infty(r, f)\right)^p \le \frac{(1-r)n\varepsilon}{\delta} + \frac{\varepsilon}{2}. \tag{82}$$

Choose a sequence $\{\varepsilon_k\}$ of positive numbers such that $\varepsilon_k \downarrow 0$. For each $k \in \mathbb{N}$, let $r_k > 0$ be a number such that

$$\frac{(1-r_k)n\varepsilon}{\delta_k} + \frac{\varepsilon_k}{2} < \varepsilon_k, \tag{83}$$

where $\varepsilon_k = \delta(\varepsilon_k)$ and

$$r_{k-1} < r_k < 1, \quad r_k \uparrow 1 \quad \text{as } k \longrightarrow \infty. \tag{84}$$

Put

$$\omega_1(r) = \varepsilon_k \quad \text{for } r_k \le r < r_{k+1}, \quad k = 1, 2, \ldots. \tag{85}$$

From (82), (83), and (85) we obtain

$$\left(\log^+ M_\infty(r, f)\right)^p \le \frac{\omega_1(r)}{1-r} \quad \forall\, 0 \le r < 1. \tag{86}$$

Since

$$\omega_1(r) \longrightarrow 0 \quad \text{as } r \longrightarrow 1, \tag{87}$$

we conclude that there exists a continuous function $\omega_2(r)$ satisfying

$$\omega_1(r) \le \omega_2(r), \quad \omega_2(r) \downarrow 0 \quad \text{as } r \to 1. \tag{88}$$

Therefore,

$$\left(\log^+ M_\infty(r, f)\right)^p \le \frac{\omega_2(r)}{1-r} \quad \text{for each } 0 \le r < 1, \tag{89}$$

whence by setting

$$\omega(r) = \left(\omega_2(r)\right)^{1/p} \quad \text{for each } 0 \le r < 1, \tag{90}$$

we obtain

$$M_\infty(r, f) \le \exp\left(\frac{\omega(r)}{(1-r)^{1/p}}\right) \quad \forall f \in L. \tag{91}$$

This concludes the proof. □

Remark 23. The condition of Theorem 22 is not a sufficient condition for a set $L \subset M^p$ to be bounded. To show this, define

$$f_n(z) = a_n z^n, \qquad a_n = \exp\left(\lambda_n n^{1/(p+1)}\right), \tag{92}$$

where

$$\lambda_n = n^{-1/2(p+1)}. \quad (93)$$

Then as in the proof of Lemma 1 in [21] it is easy to verify that the set $L = \{f_n\} \subset M^p$ satisfies the condition of Theorem 22. Since

$$\log\left|f_n^*\left(e^{i\theta}\right)\right| = n^{1/2(p+1)}, \quad (94)$$

we see that L is not bounded in M^p.

Theorem 24. *There exist bounded subsets of M^p that are not relatively compact.*

Proof. Define a sequence $\{h_n\}$ of functions on $[0, 2\pi]$ as

$$h_n(t) = 1 + \sin(nt), \quad t \in [0, 2\pi], \quad (95)$$

and set

$$f_n(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}h_n(t)\,dt\right) = \exp\left(1 - iz^n\right), \quad z \in \mathbb{D}. \quad (96)$$

Obviously, $\{f_n\} \subset N^p$ and for each measurable set $E \subset \mathbb{T}$ we have

$$\int_0^{2\pi} h_n(t)\,dt = 2\pi, \qquad 0 \le \int_E h_n(t)\,dt \le 2|E|, \quad (97)$$

where $|E|$ denotes the Lebesgue measure of E. From this and Theorem 19, we see that the set $L = \{f_n\}$ is bounded in N^p.

Now suppose that E is relatively compact. This means that there exists a subsequence $\{f_{nk}\}$ of $\{f_n\}$ and a function $f \in N^p$ such that

$$d_p\left(f_{nk}, f\right) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \quad (98)$$

and thus

$$f_{nk}(z) \longrightarrow f(z), \quad \text{uniformly on each closed disk } |z| \le r < 1. \quad (99)$$

Therefore, by (96), it follows that $f(z) \equiv e$ on $\mathbb{D}$. On the other hand, from (98), it follows that

$$f_{nk}^*\left(e^{i\theta}\right) \longrightarrow f^*\left(e^{i\theta}\right) \quad \text{in measure on } \mathbb{T}. \quad (100)$$

Therefore,

$$\log^+\left|f_{nk}^*\left(e^{i\theta}\right)\right| = 1 + \sin\left(n_k\theta\right) \longrightarrow \log^+\left|f^*\left(e^{i\theta}\right)\right| = 1 \quad \text{in measure on } \mathbb{T}. \quad (101)$$

This contradiction shows that L is not relatively compact in N^p. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] H. O. Kim, "On closed maximal ideals of M," *Proceedings of the Japan Academy A*, vol. 62, pp. 343–346, 1986.

[2] H. O. Kim, "On an F-algebra of holomorphic functions," *Canadian Journal of Mathematics*, vol. 40, pp. 718–741, 1988.

[3] I. I. Privalov, *Boundary Properties of Analytic Functions*, Izdat Moskovskogo Universiteta, Moscow, Russia, 1941.

[4] R. Meštrović and Ž. Pavićević, "Remarks on some classes of holomorphic functions," *Mathematica Montisnigri*, vol. 6, pp. 27–37, 1996.

[5] N. Mochizuki, "Algebras of holomorphic functions between H^p and N_*," *Proceedings of the American Mathematical Society*, vol. 105, pp. 898–902, 1989.

[6] M. Stoll, "Mean growth and Taylor coeffecients of some topological algebras of analytic functions," *Annales Polonici Mathematici*, vol. 35, pp. 139–158, 1977.

[7] C. M. Eoff, "Fréchet envelopes of certain algebras of analytic functions," *Michigan Mathematical Journal*, vol. 35, pp. 413–426, 1988.

[8] C. M. Eoff, "A representation of N_α^+ as a union of weighted Hardy spaces," *Complex Variables, Theory and Application*, vol. 23, pp. 189–199, 1993.

[9] Y. Iida and N. Mochizuki, "Isometries of some F-algebras of holomorphic functions," *Archiv der Mathematik*, vol. 71, no. 4, pp. 297–300, 1998.

[10] R. Meštrović, *Topological and F-algebras of holomorphic functions [Ph.D. thesis]*, University of Montenegro, Podgorica, Montenegro, 1999.

[11] R. Meštrović and Ž. Pavićević, "Topologies on some subclasses of the Smirnov class," *Acta Scientiarum Mathematicarum*, vol. 69, pp. 99–108, 2003.

[12] R. Meštrović and Ž. Pavićević, "Weakly dense ideals in Privalov spaces of holomorphic functions," *Journal of the Korean Mathematical Society*, vol. 48, no. 2, pp. 397–420, 2011.

[13] R. Meštrović and A. V. Subbotin, "Multipliers and linear functionals for Privalov spaces of holomorphic functions in the disc," *Doklady Akademii Nauk*, vol. 365, no. 4, pp. 452–454, 1999 (Russian).

[14] R. Meštrović and J. Šušić, "Interpolation in the spaces $N^p(1 < p < 1)$," *Filomat*, vol. 27, pp. 293–301, 2013.

[15] Y. Matsugu, "Invariant subspaces of the Privalov spaces," *Far East Journal of Mathematical Sciences*, vol. 2, no. 4, pp. 633–643, 2000.

[16] P. L. Duren, *Theory of* H^p *Spaces*, Academic Press, New York, NY, USA, 1970.

[17] N. Yanagihara, "Multipliers and linear functionals for the class N^+," *Transactions of the American Mathematical Society*, vol. 180, pp. 449–461, 1973.

[18] V. I. Gavrilov and V. S. Zaharyan, "Conformal invariance and the characteristic property of boundary values of functions in the class M," *Doklady Akademii Nauk Armenii*, vol. 93, no. 3, pp. 105–109, 1992 (Russian).

[19] M. Nawrocky, "On the Smirnov class defined by the maximal function," *Canadian Mathematical Bulletin*, vol. 45, pp. 265–271, 2002.

[20] N. Yanagihara, "The containing Fréchet space for the class N^+," *Duke Mathematical Journal*, vol. 40, pp. 93–103, 1973.

[21] N. Yanagihara, "Bounded subsets of some spaces of holomorphic functions," *Scientific Papers of the College of General Education, University of Tokyo*, vol. 23, pp. 19–28, 1973.

[22] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, vol. 7 of *Pure and Applied Mathematics*, Wiley-Interscience, New York, NY, USA, 1958.

[23] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, NY, USA, 1973.