

Research Article

New Proofs of Some q -Summation and q -Transformation Formulas

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We obtain an expectation formula and give the probabilistic proofs of some summation and transformation formulas of q -series based on our expectation formula. Although these formulas in themselves are not the probability results, the proofs given are based on probabilistic concepts.

1. Introduction

The probabilistic method is an important tool to derive results in combinatorics, theory of numbers, and other fields (see [1–15]). There have been many applications in the basic hypergeometric series (or q -series). For example, Fulman [3] presented a probabilistic proof of Rogers-Ramanujan identity using Markov chain. Chapman [2] proved the Andrews-Gordon identity by using extended Fulman's methods. Kadell [4] gave a probabilistic proof of Ramanujan's ${}_1\psi_1$ summation based on the order statistics.

Recently, Wang [13, 14] constructed a random variable X and introduced a new probability distribution $W(x; q)$:

$$\begin{aligned} P(X = x^n q^k) &= p_{n,k}(x; q) \\ &=: \frac{(-x)^n q^k (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_\infty}{(q, q/x, x; q)_\infty}, \end{aligned} \quad (1)$$

where

$$p_{n,k}(x; q) > 0, \quad \sum p_{n,k}(x; q) = 1, \quad (2)$$

$$x < 0, \quad 0 < q < 1, \quad n = 0, 1, \quad k = 0, 1, 2, \dots$$

By applying the above probability distribution, Wang proved the q -binomial theorem and q -Gauss summation formula

and also obtained some new summation formulas and transformation formulas.

One of the most important concepts in probability theory is that of the expectation of a random variable. If X is a discrete random variable having a probability mass function $p(x)$, then the *expectation*, or the *expected value*, or the *expectation operator* of X , denoted by $\mathbb{E}[X]$, is defined by (e.g., [9, page 125])

$$\mathbb{E}[X] = \sum_{p(x)>0} x p(x). \quad (3)$$

In the following section we introduce some notations, definitions, and formulas of q -series. Throughout this paper we suppose $q \in \mathbb{C}$, $|q| < 1$.

The q -shifted factorials are defined by

$$\begin{aligned} (a; q)_0 &= 1, & (a; q)_n &= \prod_{k=0}^{n-1} (1 - aq^k), \\ (a; q)_\infty &= \prod_{k=0}^{\infty} (1 - aq^k), & n &\geq 1. \end{aligned} \quad (4)$$

Clearly,

$$\begin{aligned} (a; q)_n &= \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \\ (a^2; q^2)_n &= (a; q)_n (-a; q)_n. \end{aligned} \quad (5)$$

The following are compact notations for the multiple q -shifted factorials:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \quad (6)$$

$$(a_1, a_2, \dots, a_m; q)_{\infty} = (a_1; q)_{\infty} (a_2; q)_{\infty} \cdots (a_m; q)_{\infty}.$$

The basic hypergeometric series or q -series ${}_r\phi_s$ are defined by (see [16, 17])

$$\begin{aligned} {}_r\phi_s &\left(\begin{array}{c} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{array}; q, z \right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, b_2, \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n. \end{aligned} \quad (7)$$

Heine introduced the ${}_{r+1}\phi_r$ basic hypergeometric series which is defined by

$${}_{r+1}\phi_r \left(\begin{array}{c} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{array}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, b_2, \dots, b_r; q)_n} z^n. \quad (8)$$

Jackson defined the q -integral (see [17, 18]):

$$\int_0^d f(t) d_q t = d(1-q) \sum_{n=0}^{\infty} f(dq^n) q^n, \quad (9)$$

$$\int_c^d f(t) d_q t = \int_0^d f(t) d_q t - \int_0^c f(t) d_q t. \quad (10)$$

The following is the Andrews-Askey integral (see [19]) which can be derived from Ramanujan's ${}_1\psi_1$:

$$\int_c^d \frac{(qt/c, qt/d; q)_{\infty}}{(at, bt; q)_{\infty}} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_{\infty}}{(ac, ad, bc, bd; q)_{\infty}}, \quad (11)$$

provided that there are no zero factors in the denominator of the integrals. Recently, Liu and Luo [20] further generalized the above Andrews-Askey integral in the following more general form.

Lemma 1 (see [21, page 5. (2.5)] [20, Theorem 1]). *One has*

$$\begin{aligned} &\int_a^b \frac{(qt/a, qt/b, ct; q)_{\infty}}{(et, ft, ht; q)_{\infty}} d_q t \\ &= \frac{b(1-q)(q, bq/a, a/b, bc, abef; q)_{\infty}}{(ae, af, be, bf, bh; q)_{\infty}} {}_3\phi_2 \\ &\quad \times \left(\begin{array}{c} be, bf, \frac{c}{h} \\ abef, bc \end{array}; q, ah \right), \end{aligned} \quad (12)$$

provided $|ah| < 1$ and $|q| < 1$, provided that there are no zero factors in the denominator of the integrals.

Lemma 2 (see [21, page 5. (2.7)]). *One has*

$$\begin{aligned} &\int_a^b \frac{(qt/a, qt/b, ct; q)_{\infty}}{(et, ht; q)_{\infty}} d_q t \\ &= \frac{b(1-q)(q, bq/a, a/b, bc; q)_{\infty}}{(ae, be, bh; q)_{\infty}} {}_2\phi_1 \left(\begin{array}{c} be, \frac{c}{h} \\ bc \end{array}; q, ah \right), \end{aligned} \quad (13)$$

provided $|ah| < 1$ and $|q| < 1$, provided that there are no zero factors in the denominator of the integrals.

The aim of the present paper is to give an expectation formula and introduce some probabilistic proofs of the corresponding summation and transformation formulas of q -series based on an expectation formula. In Section 2 we give an expectation formula of the random variables $(dX; q)_{\infty}/(aX, bX, cX; q)_{\infty}$. In Section 3 we show the probabilistic proofs of transformation formulas of ${}_3\phi_2$. In Section 4 we give probabilistic proof of Heine's transformations and Jackson's transformations. In Section 5 we give probabilistic proof of some formulas of q -series, for example, q -binomial theorem, q -Chu-Vandermonde sum formulas, q -Gauss sum formula, q -Kummer sum formula, Bailey sum formula, and so forth.

2. Main Theorem

In this section we obtain the expectation formulas of some random variables which are very useful to prove the summation and transformation formulas of q -series.

Theorem 3. *Let X denote a random variable with probability distribution $W(x; q)$, $-1 < x < 0$. Then one has*

$$\begin{aligned} &\mathbb{E} \left[\frac{(dX; q)_{\infty}}{(aX, bX, cX; q)_{\infty}} \right] \\ &= \frac{(d, abx; q)_{\infty}}{(a, ax, b, bx, c; q)_{\infty}} {}_3\phi_2 \left(\begin{array}{c} a, b, \frac{d}{c} \\ abx, d \end{array}; q, cx \right), \end{aligned} \quad (14)$$

provided that $\max(|a|, |b|, |c|, |d|) < 1$, $|cx| < 1$, and $|q| < 1$.

Proof. A random variable X has the distribution $W(x; q)$. From definitions (9) we have

$$\begin{aligned} &\int_0^1 \frac{(qt/x, qt, dt; q)_{\infty}}{(at, bt, ct; q)_{\infty}} d_q t \\ &= (1-q) \sum_{k=0}^{\infty} \frac{(q^{k+1}/x, q^{k+1}, dq^k; q)_{\infty} q^k}{(aq^k, bq^k, cq^k; q)_{\infty}}, \end{aligned} \quad (15)$$

$$\begin{aligned} &\int_0^x \frac{(qt/x, qt, dt; q)_{\infty}}{(at, bt, ct; q)_{\infty}} d_q t \\ &= x(1-q) \sum_{k=0}^{\infty} \frac{(q^{k+1}, xq^{k+1}, dxq^k; q)_{\infty} q^k}{(axq^k, bxq^k, cxq^k; q)_{\infty}}. \end{aligned}$$

From definitions (10) and combining (15) we have

$$\begin{aligned} & \int_x^1 \frac{(qt/x, qt, dt; q)_\infty}{(at, bt, ct; q)_\infty} d_q t \\ &= (1-q) \sum_{k=0}^{\infty} \frac{(q^{k+1}/x, q^{k+1}, dq^k; q)_\infty q^k}{(aq^k, bq^k, cq^k; q)_\infty} \\ & \quad - x(1-q) \sum_{k=0}^{\infty} \frac{(q^{k+1}, xq^{k+1}, dxq^k; q)_\infty q^k}{(axq^k, bxq^k, cxq^k; q)_\infty}. \end{aligned} \quad (16)$$

By using the probability distribution $W(x; q)$ and noting (16) and (12) of Lemma 1, we calculate the expectation of the random variable $(dX; q)_\infty/(aX, bX, cX; q)_\infty$ as follows:

$$\begin{aligned} & \mathbb{E} \left[\frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] \\ &= \sum_{n=0}^1 \sum_{k=0}^{\infty} \frac{(-x)^n (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty} \\ & \quad \times \frac{(dx^n q^k; q)_\infty}{(ax^n q^k, bx^n q^k, cx^n q^k; q)_\infty} \\ &= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \\ & \quad \times \left((1-q) \sum_{k=0}^{\infty} \frac{(q^{k+1}/x, q^{k+1}, dq^k; q)_\infty q^k}{(aq^k, bq^k, cq^k; q)_\infty} \right. \\ & \quad \left. - x(1-q) \sum_{k=0}^{\infty} \frac{(q^{k+1}, xq^{k+1}, dxq^k; q)_\infty q^k}{(axq^k, bxq^k, cxq^k; q)_\infty} \right) \\ &= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \int_x^1 \frac{(qt/x, qt, dt; q)_\infty}{(at, bt, ct; q)_\infty} d_q t \\ &= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \frac{(1-q)(q, q/x, x, d, abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} {}_3\phi_2 \cdot \\ & \quad \times \left(\begin{matrix} a, b, \frac{d}{c} \\ abx, d \end{matrix} ; q, cx \right). \end{aligned} \quad (17)$$

Hence, we obtain

$$\begin{aligned} & \mathbb{E} \left[\frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] \\ &= \frac{(d, abx; q)_\infty}{(ax, a, bx, b, c; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, b, \frac{d}{c} \\ abx, d \end{matrix} ; q, cx \right). \end{aligned} \quad (18)$$

The proof is complete. \square

Theorem 4. Let X denote a random variable with probability distribution $W(x; q)$, $-1 < x < 0$. Then one has

$$\mathbb{E} \left[\frac{(dX; q)_\infty}{(aX, bX; q)_\infty} \right] = \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} {}_2\phi_2 \left(\begin{matrix} a, b \\ abx, d \end{matrix} ; q, dx \right), \quad (19)$$

provided that $\max(|a|, |b|, |d|) < 1$, $|dx| < 1$, and $|q| < 1$.

Proof. By (7) and (8) we have

$$\begin{aligned} & {}_3\phi_2 \left(\begin{matrix} a, b, \frac{d}{c} \\ abx, d \end{matrix} ; q, cx \right) \\ &= \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n (d/c; q)_n (cx)^n}{(q; q)_n (abx; q)_n (d; q)_n} \\ &= \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n x^n}{(q; q)_n (abx; q)_n (d; q)_n} \left(\frac{d}{c}; q \right)_n c^n. \end{aligned} \quad (20)$$

By (4) we have

$$\begin{aligned} \left(\frac{d}{c}; q \right)_n c^n &= \left(1 - \frac{d}{c} \right) \left(1 - \frac{d}{c} q \right) \cdots \left(1 - \frac{d}{c} q^{n-1} \right) c^n \\ &= (c-d)(c-dq)\cdots(c-dq^{n-1}). \end{aligned} \quad (21)$$

Substituting (20) and (21) into the right-hand side of (14), we obtain

$$\begin{aligned} & \mathbb{E} \left[\frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] \\ &= \frac{(d, abx; q)_\infty}{(a, ax, b, bx, c; q)_\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n x^n}{(q; q)_n (abx; q)_n (d; q)_n} \\ & \quad \times (c-d)(c-dq)\cdots(c-dq^{n-1}). \end{aligned} \quad (22)$$

Next, let us replace c by λc , respectively, and let $\lambda \rightarrow 0$ in (22); we get

$$\begin{aligned} & \mathbb{E} \left[\frac{(dX; q)_\infty}{(aX, bX; q)_\infty} \right] \\ &= \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n x^n}{(q; q)_n (abx; q)_n (d; q)_n} \\ & \quad \times (-d)(-dq)\cdots(-dq^{n-1}) \\ &= \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n x^n}{(q; q)_n (abx; q)_n (d; q)_n} \\ & \quad \times d^n (-1)^n q^{n(n-1)/2} \\ &= \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n (dx)^n}{(q; q)_n (abx; q)_n (d; q)_n} (-1)^n q^{\binom{n}{2}} \\ &= \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} {}_2\phi_2 \left(\begin{matrix} a, b \\ abx, d \end{matrix} ; q, dx \right). \end{aligned} \quad (23)$$

The proof is complete. \square

\square

Theorem 5. Let X denote a random variable with probability distribution $W(x; q)$, $-1 < x < 0$. Then one has

$$\begin{aligned} & \mathbb{E} \left[\frac{(cxX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] \\ &= \frac{(cx, abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, b, x \\ abx, cx \end{matrix}; q, cx \right), \end{aligned} \quad (24)$$

provided that $\max(|a|, |b|, |c|) < 1$, $|cx| < 1$, and $|q| < 1$.

Proof. Letting $d = cx$ in (14) of Theorem 3, we obtain (24). \square

Corollary 6 (see [13, page 463, Theorem 1]). Let X denote a random variable with probability distribution $W(x; q)$, $-1 < x < 0$. Then one has

$$\begin{aligned} & \mathbb{E} \left[\frac{(cxX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] \\ &= \frac{(acx, bcx, x; q)_\infty}{(a, ax, b, bx, c, cx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} ab, c, cx \\ acx, bcx \end{matrix}; q, x \right), \end{aligned} \quad (25)$$

provided that $\max(|a|, |b|, |c|) < 1$.

Proof. Using (24) of Theorem 5, we deduce

$$\begin{aligned} & \mathbb{E} \left[\frac{(cxX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] \\ &= \frac{(cx, abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, b, x \\ abx, cx \end{matrix}; q, cx \right). \end{aligned} \quad (26)$$

Using (31) of Theorem 8, we have

$$\begin{aligned} & {}_3\phi_2 \left(\begin{matrix} a, b, x \\ abx, cx \end{matrix}; q, cx \right) \\ &= {}_3\phi_2 \left(\begin{matrix} a, x, b \\ abx, cx \end{matrix}; q, cx \right) \\ &= \frac{(x, bcx, acx; q)_\infty}{(abx, cx, cx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} ab, c, cx \\ bcx, acx \end{matrix}; q, x \right). \end{aligned} \quad (27)$$

Substituting (27) into the right-hand sides of (26), we have

$$\begin{aligned} & \mathbb{E} \left[\frac{(cxX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] \\ &= \frac{(cx, abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} \frac{(x, bcx, acx; q)_\infty}{(abx, cx, cx; q)_\infty} {}_3\phi_2 \\ & \quad \times \left(\begin{matrix} ab, c, cx \\ bcx, acx \end{matrix}; q, x \right) \\ &= \frac{(acx, bcx, x; q)_\infty}{(a, ax, b, bx, c, cx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} ab, c, cx \\ acx, bcx \end{matrix}; q, x \right). \end{aligned} \quad (28)$$

The proof is complete. \square

Corollary 7 (see [14, page 245, Lemma 2.4]). Let X denote a random variable with probability distribution $W(x; q)$, $-1 < x < 0$. Then one has

$$\mathbb{E} \left[\frac{1}{(aX, bX; q)_\infty} \right] = \frac{(abx; q)_\infty}{(ax, bx, a, b; q)_\infty}, \quad (29)$$

provided that $\max(|a|, |b|) < 1$.

Proof. Letting $d = c$ or $d = a = c$ in (14) of Theorem 3, then we have

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{(aX, bX; q)_\infty} \right] = \frac{(abx; q)_\infty}{(ax, bx, a, b; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, b, 1 \\ abx, d \end{matrix}; q, cx \right) \\ &= \frac{(abx; q)_\infty}{(ax, bx, a, b; q)_\infty}. \end{aligned} \quad (30)$$

The proof is complete. \square

3. Probabilistic Proofs of Transformation Formulas of ${}_3\phi_2$

Sears' ${}_3\phi_2$ transformation formula is widely applied to the special functions. In this section we will introduce probabilistic proofs of transformation of ${}_3\phi_2$.

Theorem 8 (see [17, page 359. III. 9, III. 10]). One has

$$\begin{aligned} & {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc} \right) \\ &= \frac{(e/a, de/bc; q)_\infty}{(e, de/abc; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, \frac{d}{b}, \frac{d}{c} \\ d, \frac{de}{bc} \end{matrix}; q, \frac{e}{a} \right) \\ &= \frac{(b, de/ab, de/bc; q)_\infty}{(d, e, de/abc; q)_\infty} {}_3\phi_2 \left(\begin{matrix} \frac{d}{b}, \frac{e}{b}, \frac{de}{qbc} \\ \frac{de}{ab}, \frac{de}{bc} \end{matrix}; q, b \right). \end{aligned} \quad (31)$$

Proof. Interchanging b and c in (14), then we have

$$\begin{aligned} & \mathbb{E} \left[\frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] \\ &= \frac{(d, acx; q)_\infty}{(ax, a, cx, c, b; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, c, \frac{d}{b} \\ acx, d \end{matrix}; q, bx \right). \end{aligned} \quad (33)$$

Interchanging a and c in (14), then we have

$$\begin{aligned} & \mathbb{E} \left[\frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] \\ &= \frac{(d, bcx; q)_\infty}{(bx, b, cx, c, a; q)_\infty} {}_3\phi_2 \left(\begin{matrix} c, b, \frac{d}{a} \\ bcx, d \end{matrix}; q, ax \right). \end{aligned} \quad (34)$$

By (14) and (33), we obtain

$$\begin{aligned} {}_3\phi_2 \left(\begin{matrix} a, b, \frac{d}{c} \\ abx, d \end{matrix}; q, cx \right) \\ = \frac{(bx, acx; q)_\infty}{(cx, abx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, c, \frac{d}{b} \\ acx, d \end{matrix}; q, bx \right), \end{aligned} \quad (35)$$

and, replacing $(a, b, d/c, d, abx)$ by (a, b, c, d, e) in (35), we obtain a ${}_3\phi_2$ transformation formula

$$\begin{aligned} {}_3\phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc} \right) \\ = \frac{(e/a, de/bc; q)_\infty}{(e, de/abc; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, \frac{d}{b}, \frac{d}{c} \\ d, \frac{de}{bc} \end{matrix}; q, \frac{e}{a} \right). \end{aligned} \quad (36)$$

By (14) and (34) and then replacing $(b, a, d/c, d, abx)$ by (a, b, c, d, e) , we obtain (31).

By (33) and (34), we have

$$\begin{aligned} {}_3\phi_2 \left(\begin{matrix} c, b, \frac{d}{a} \\ bcx, d \end{matrix}; q, ax \right) \\ = \frac{(bx, acx; q)_\infty}{(ax, bcx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, c, \frac{d}{b} \\ acx, d \end{matrix}; q, bx \right), \end{aligned} \quad (37)$$

and, replacing $(c, b, d/a, d, bcx)$ by (a, b, c, d, e) in (37), we obtain (32). The proof is complete. \square

4. Probabilistic Proof of Heine and Jackson's Transformations

Heine [22] derived transformation formulas for ${}_2\phi_1$ and also proved Euler's transformation formula. A basic hypergeometric representation for a given function is by no means unique. There are groups of transformation between various hypergeometric representations of the same function. We will first prove the classical Heine's transformation formula which will be useful in proving many other formulas. In this section we give the probabilistic proofs of Heine and Jackson's transformations.

Theorem 9 (see [17, page 359, III. 1, III. 2, III. 3]). *Heine's transformation formulas for ${}_2\phi_1$ are*

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, z \right) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} \frac{c}{b}, z \\ az \end{matrix}; q, b \right) \quad (38)$$

$$= \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} abz \\ \frac{c}{bz}, b \end{matrix}; q, \frac{c}{b} \right) \quad (39)$$

$$= \frac{(abz/c; q)_\infty}{(z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} \frac{c}{a}, \frac{c}{b} \\ \frac{c}{a}, \frac{abz}{c} \end{matrix}; q, \frac{abz}{c} \right). \quad (40)$$

Proof. Comparing (24) of Theorem 5 and (25) of Corollary 6, we obtain

$$\begin{aligned} & \frac{(acx, bcx, x; q)_\infty}{(a, ax, b, bx, c, cx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} ab, c, cx \\ acx, bcx \end{matrix}; q, x \right) \\ &= \frac{(cx, abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, b, x \\ abx, cx \end{matrix}; q, cx \right), \end{aligned} \quad (41)$$

or, equivalently, that

$$\begin{aligned} & {}_3\phi_2 \left(\begin{matrix} ab, c, cx \\ acx, bcx \end{matrix}; q, x \right) \\ &= \frac{(cx, abx, cx; q)_\infty}{(acx, bcx, x; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, b, x \\ abx, cx \end{matrix}; q, cx \right). \end{aligned} \quad (42)$$

Setting $b = 0$ in (42), we have

$${}_2\phi_1 \left(\begin{matrix} c, cx \\ acx \end{matrix}; q, x \right) = \frac{(cx, cx; q)_\infty}{(acx, x; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, x \\ cx \end{matrix}; q, cx \right). \quad (43)$$

Replacing (c, cx, acx, x) by (a, b, c, z) in (43), we get

$$\begin{aligned} {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, z \right) &= \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} \frac{c}{b}, z \\ az \end{matrix}; q, b \right) \\ &\text{for } |b| < 1, |z| < 1, \end{aligned} \quad (44)$$

which is just (38).

Setting $d = 0$ and $a = b$ and replacing c by b in (14), we have

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{(aX, aX, bX; q)_\infty} \right] \\ &= \frac{(a^2 x; q)_\infty}{(ax, ax, a, a, b; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, a \\ a^2 x \end{matrix}; q, bx \right). \end{aligned} \quad (45)$$

Setting $d = 0$ and $a = c$ in (14) of Theorem 3, we have

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{(aX, aX, bX; q)_\infty} \right] \\ &= \frac{(abx; q)_\infty}{(ax, bx, a, a, b; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, b \\ abx \end{matrix}; q, ax \right). \end{aligned} \quad (46)$$

Comparing (45) and (46), we obtain

$${}_2\phi_1 \left(\begin{matrix} b, a \\ abx \end{matrix}; q, ax \right) = \frac{(bx, a^2 x; q)_\infty}{(abx, ax; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, a \\ a^2 x \end{matrix}; q, bx \right). \quad (47)$$

Replacing (a, b, x) by $(b, a, z/b)$, we get

$${}_2\phi_1 \left(\begin{matrix} a, b \\ az \end{matrix}; q, z \right) = \frac{(az/b, bz; q)_\infty}{(az, z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} b, b \\ bz \end{matrix}; q, \frac{az}{b} \right). \quad (48)$$

Letting $c = az$ in (48) gives

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, z \right) = \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} abz \\ bz \end{matrix}; q, \frac{c}{b} \right). \quad (49)$$

We get (39). From (39) we can deduce (40). \square

Jackson's transformations formula is an important formula in basic hypergeometric series, and now we give a probabilistic proof of Jackson's transformation formulas for ${}_2\phi_1$ and ${}_2\phi_2$.

Theorem 10 (see [17, page 359, III.4]). *Jackson's transformations of ${}_2\phi_1$, ${}_2\phi_2$ series are*

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, z \right) = \frac{(az; q)_\infty}{(z; q)_\infty} {}_2\phi_2 \left(\begin{matrix} a, \frac{c}{b} \\ c, az \end{matrix}; q, bz \right). \quad (50)$$

Proof. This includes employing two different forms of $\mathbb{E}[(dX; q)_\infty / (aX, bX; q)_\infty]$.

Letting $b = 0$ in (14) of Theorem 3 and then replacing c by b , we get

$$\mathbb{E} \left[\frac{(dX; q)_\infty}{(aX, bX; q)_\infty} \right] = \frac{(d; q)_\infty}{(a, ax, b; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, \frac{d}{b} \\ d \end{matrix}; q, bx \right). \quad (51)$$

Comparing (51) and (19) of Theorem 4 gives

$$\begin{aligned} & \frac{(d; q)_\infty}{(a, ax, b; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, \frac{d}{b} \\ d \end{matrix}; q, bx \right) \\ &= \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} {}_2\phi_2 \left(\begin{matrix} a, b \\ abx, d \end{matrix}; q, dx \right). \end{aligned} \quad (52)$$

Then we obtain

$${}_2\phi_1 \left(\begin{matrix} a, \frac{d}{b} \\ d \end{matrix}; q, bx \right) = \frac{(abx; q)_\infty}{(bx; q)_\infty} {}_2\phi_2 \left(\begin{matrix} a, b \\ abx, d \end{matrix}; q, dx \right). \quad (53)$$

Replacing $(a, d/b, d, bx)$ by (a, b, c, z) gives

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, z \right) = \frac{(az; q)_\infty}{(z; q)_\infty} {}_2\phi_2 \left(\begin{matrix} a, \frac{c}{b} \\ c, az \end{matrix}; q, bz \right). \quad (54)$$

This completes the proof. \square

5. Probabilistic Proofs of Some Formulas of q -Series

The q -binomial theorem is an important mathematical result which has been widely applied in the special functions, physics, quantum algebra, and quantum statistics. The q -binomial theorem was derived by Cauchy [23], Heine [22],

and Jacobi [24] concerning the nonterminating form. There are many proofs of the q -binomial theorem to show the corresponding references; for example, a better and simpler proof, by using the method of the finite difference, was obtained by Askey (see [25]); a nice proof of the q -binomial theorem based on combinatorial considerations was given by Joichi and Stanton (see [26]). In 1847, Heine [22] derived a q -analogue of Gauss's summation formula which is important in q -series. Joichi and Stanton [26] gave a bijective proof of the q -Gauss summation formula based on combinatorial considerations. Rahman and Suslov [27] used the method of the first order linear difference equations to prove the q -Gauss summation formula. By analytic continuation, the terminating case, when $a = q^{-n}$, reduces to q -analogues of Vandermonde's formula. Bailey and Daum independently discovered the q -Kummer summation formula.

In this section we will introduce probabilistic proof of some formulas of q -series, for example, q -binomial theorem, q -Chu-Vandermonde, q -Gauss summation formula and q -Kummer summation formula, and so forth.

Theorem 11 (see [16, page 488, Theorem 10.2.1] [17, page 354, II. 3]). *The q -binomial theorem is*

$${}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty} \quad (55)$$

for $|z| < 1$, $|q| < 1$.

Proof. Below we give two proofs of (55).

Setting $d = a = 0$ and replacing b and c by a and b in (14), we obtain

$$\mathbb{E} \left[\frac{1}{(ax, bx; q)_\infty} \right] = \frac{1}{(ax, a, b; q)_\infty} {}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, bx \right). \quad (56)$$

Comparing (56) and (29) of Corollary 7, we have

$$\frac{1}{(ax, a, b; q)_\infty} {}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, bx \right) = \frac{(abx; q)_\infty}{(ax, bx, a, b; q)_\infty}. \quad (57)$$

Then we obtain

$${}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, bx \right) = \frac{(abx; q)_\infty}{(bx; q)_\infty}. \quad (58)$$

Replacing bx by z , we can get

$${}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, z \right) = \frac{(az; q)_\infty}{(z; q)_\infty}; \quad (59)$$

that is,

$${}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty} \quad \text{for } |z| < 1. \quad (60)$$

Another proof of the q -binomial theorem is as follows.

Setting $d = a = 0$ and $b = c$ and replacing b by a in (14), we obtain

$$\mathbb{E} \left[\frac{1}{(aX; q)_\infty^2} \right] = \frac{1}{(ax, a, a; q)_\infty} {}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, ax \right). \quad (61)$$

Letting $a = b = c = d$ or $a = b$ and $d = c$ in (14) of Theorem 3, we obtain

$$\begin{aligned} \mathbb{E} \left[\frac{1}{(aX; q)_\infty^2} \right] &= \frac{(a, a^2x; q)_\infty}{(ax, ax, a, a; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, a, 1 \\ a^2x, a \end{matrix}; q, ax \right) \\ &= \frac{(a, a^2x; q)_\infty}{(ax, ax, a, a; q)_\infty} \\ &= \frac{(a^2x; q)_\infty}{(ax, a; q)_\infty^2}. \end{aligned} \quad (62)$$

Comparing (61) and (62) gives

$$\frac{1}{(ax, a, a; q)_\infty} {}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, ax \right) = \frac{(a^2x; q)_\infty}{(ax, a; q)_\infty^2}. \quad (63)$$

Then we obtain

$${}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, ax \right) = \frac{(a^2x; q)_\infty}{(ax; q)_\infty}. \quad (64)$$

Replacing ax by z gives

$${}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, z \right) = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad (65)$$

that is,

$${}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty} \quad \text{for } |z| < 1. \quad (66)$$

This proof is complete. \square

Theorem 12 (see [17, page 354, II. 7]). *The q -Chu-Vandermonde sums are*

$${}_2\phi_1 \left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, \frac{cq^n}{a} \right) = \frac{(c/a; q)_n}{(c; q)_n}. \quad (67)$$

Proof. The below are two proofs of the q -Chu-Vandermonde.

(i) First proof: setting $d = a = b$ and replacing c by b in (14), we have

$$\mathbb{E} \left[\frac{1}{(aX, bX; q)_\infty} \right] = \frac{(a^2x; q)_\infty}{(ax, ax, a, b; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, \frac{a}{b} \\ a^2x \end{matrix}; q, bx \right). \quad (68)$$

Replacing (a, b, x) by $(a, aq^n, c/a^2)$ in (68), then we have

$$\begin{aligned} P \left(Y = \left(\frac{c}{a^2} \right)^n q^k \right) \\ = p_{n,k} \left(\frac{c}{a^2}; q \right) \\ := \frac{(-c/a^2)^n q^k ((c/a^2)^{n-1} q^{k+1}, (c/a^2)^n q^{k+1}; q)_\infty}{(q, a^2q/c, c/a^2; q)_\infty}, \end{aligned} \quad (69)$$

where

$$\begin{aligned} p_{n,k} \left(\frac{c}{a^2}; q \right) &> 0, \quad \sum p_{n,k} \left(\frac{c}{a^2}; q \right) = 1, \\ \frac{c}{a^2} &< 0, \quad 0 < q < 1, \quad n = 0, 1, \quad k = 0, 1, 2, \dots \end{aligned} \quad (70)$$

Hence,

$$\begin{aligned} \mathbb{E} \left[\frac{1}{(aY, aq^nY; q)_\infty} \right] \\ = \frac{(c; q)_\infty}{(c/a, c/a, a, aq^n; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, \frac{cq^n}{a} \right). \end{aligned} \quad (71)$$

By using the probability distribution $W(c/a^2; q)$ and employing Andrews-Askey q -integral (11), now we calculate the expectation of the random variables $1/(aY, aq^nY; q)_\infty$ as follows:

$$\begin{aligned} \mathbb{E} \left[\frac{1}{(aY, aq^nY; q)_\infty} \right] \\ = \sum_{n=0}^1 \sum_{k=0}^{\infty} \frac{(-c/a^2)^n q^k ((c/a^2)^{n-1} q^{k+1}, (c/a^2)^n q^{k+1}; q)_\infty}{(q, a^2q/c, c/a^2, a(c/a^2)^n q^k, (aq^n)(c/a^2)^n q^k; q)_\infty} \\ = \frac{1}{(1-q)(q, a^2q/c, c/a^2; q)_\infty} \\ \times \left((1-q) \sum_{k=0}^{\infty} \frac{(q^{k+1}/(c/a^2), q^{k+1}; q)_\infty q^k}{(aq^k, (aq^n)q^k; q)_\infty} - \left(\frac{c}{a^2} \right) (1-q) \right. \\ \left. \times \sum_{k=0}^{\infty} \frac{(q^{k+1}, (c/a^2)q^{k+1}; q)_\infty q^k}{(a(c/a^2)q^k, (aq^n)(c/a^2)q^k; q)_\infty} \right) \\ = \frac{1}{(1-q)(q, a^2q/c, c/a^2; q)_\infty} \int_{c/a^2}^1 \frac{(qt/(c/a^2), qt; q)_\infty}{(at, aq^nt; q)_\infty} d_q t \\ = \frac{1}{(1-q)(q, a^2q/c, c/a^2; q)_\infty} \frac{(1-q)(q, a^2q/c, c/a^2, cq^n; q)_\infty}{(c/a, cq^n/a, a, aq^n; q)_\infty} \\ = \frac{(cq^n; q)_\infty}{(c/a, cq^n/a, a, aq^n; q)_\infty}. \end{aligned} \quad (72)$$

Comparing (71) and (72) gives

$$\begin{aligned} & \frac{(c;q)_\infty}{(c/a, c/a, a, aq^n; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, \frac{cq^n}{a}\right) \\ &= \frac{(cq^n; q)_\infty}{(c/a, cq^n/a, a, aq^n; q)_\infty}. \end{aligned} \quad (73)$$

Then we obtain

$$\begin{aligned} & {}_2\phi_1\left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, \frac{cq^n}{a}\right) \\ &= \frac{(cq^n; q)_\infty}{(c/a, cq^n/a, a, aq^n; q)_\infty} \frac{(c/a, c/a, a, aq^n; q)_\infty}{(c; q)_\infty} \\ &= \frac{(cq^n; q)_\infty (c/a; q)_\infty}{(c; q)_\infty (cq^n/a; q)_\infty} \\ &= \frac{(c/a; q)_n}{(c; q)_n}, \end{aligned} \quad (74)$$

which is just q -Vandermonde sums (67).

(ii) Second proof: replacing (a, b, x) by $(a, aq^n, c/a^2)$ in (29), we have

$$\mathbb{E}\left[\frac{1}{(aY, aq^nY; q)_\infty}\right] = \frac{(cq^n; q)_\infty}{(c/a, cq^n/a, a, aq^n; q)_\infty}. \quad (75)$$

Comparing (71) and (75), we obtain

$$\begin{aligned} & \frac{(c;q)_\infty}{(c/a, c/a, a, aq^n; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, \frac{cq^n}{a}\right) \\ &= \frac{(cq^n; q)_\infty}{(c/a, cq^n/a, a, aq^n; q)_\infty}. \end{aligned} \quad (76)$$

Hence,

$$\begin{aligned} & {}_2\phi_1\left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, \frac{cq^n}{a}\right) = \frac{(cq^n; q)_\infty (c/a; q)_\infty}{(c; q)_\infty (cq^n/a; q)_\infty} \\ &= \frac{(c/a; q)_n}{(c; q)_n}, \end{aligned} \quad (77)$$

which is just q -Vandermonde sums (67). \square

Theorem 13 (see [16, page 522, Corollary 10.9.2] or [17, page 354, II. 8]). *The q -Gauss sum is*

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}. \quad (78)$$

Proof. Letting $d = a = b$ and replacing c by b in (14), we obtain

$$\mathbb{E}\left[\frac{1}{(aX, bX; q)_\infty}\right] = \frac{(a^2x; q)_\infty}{(ax, ax, a, b; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, \frac{a}{b} \\ a^2x \end{matrix}; q, bx\right). \quad (79)$$

Comparing (29) and (79) gives

$$\frac{(a^2x; q)_\infty}{(ax, ax, a, b; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, \frac{a}{b} \\ a^2x \end{matrix}; q, bx\right) = \frac{(abx; q)_\infty}{(ax, bx, a, b; q)_\infty}, \quad (80)$$

hence we get

$${}_2\phi_1\left(\begin{matrix} a, \frac{a}{b} \\ a^2x \end{matrix}; q, bx\right) = \frac{(abx, ax; q)_\infty}{(a^2x, bx; q)_\infty}. \quad (81)$$

Replacing $(a, a/b, a^2x)$ by (a, b, c) in the above formula, we obtain

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad (82)$$

which is just the q -Gauss sum (78). \square

Theorem 14 (see [17, page 354, II. 9]). *The q -Kummer sum formula is*

$${}_2\phi_1\left(\begin{matrix} a, b \\ \frac{aq}{b} \end{matrix}; q, -\frac{q}{b}\right) = \frac{(-q; q)_\infty (aq, aq^2/b^2; q^2)_\infty}{(-q/b, aq/b; q)_\infty}. \quad (83)$$

Proof. Letting $b = 0$ in (14) and then replacing c by b , we have

$$\mathbb{E}\left[\frac{(dX; q)_\infty}{(aX, bX; q)_\infty}\right] = \frac{(d; q)_\infty}{(a, ax, b; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, \frac{d}{b} \\ d \end{matrix}; q, bx\right). \quad (84)$$

Replacing (a, b, d, x) by $(a, aq/b^2, aq/b, -b/a)$ in (84), we write

$$\begin{aligned} & P\left(Z = \left(-\frac{b}{a}\right)^n q^k\right) \\ &= p_{n,k}\left(-\frac{b}{a}; q\right) \\ &=: \frac{(b/a)^n q^k ((-b/a)^{n-1} q^{k+1}, (-b/a)^n q^{k+1}; q)_\infty}{(q, -aq/b, -b/a; q)_\infty}, \end{aligned} \quad (85)$$

where

$$p_{n,k}\left(-\frac{b}{a}; q\right) > 0, \quad \sum p_{n,k}\left(-\frac{b}{a}; q\right) = 1, \quad (86)$$

$$-\frac{b}{a} < 0, \quad 0 < q < 1, \quad n = 0, 1, \quad k = 0, 1, 2, \dots$$

Hence, we obtain

$$\begin{aligned} & \mathbb{E}\left[\frac{((aq/b)Z; q)_\infty}{(aZ, (aq/b^2)Z; q)_\infty}\right] \\ &= \frac{(aq/b; q)_\infty}{(a, -b, aq/b^2; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, b \\ \frac{aq}{b} \end{matrix}; q, -\frac{q}{b}\right). \end{aligned} \quad (87)$$

By using the probability distribution $W(-b/a; q)$ and Lemma 2, we calculate the expectation of the random variables $((aq/b)Z; q)_\infty / (aZ, (aq/b^2)Z; q)_\infty$ as follows:

$$\begin{aligned}
 & \mathbb{E} \left[\frac{((aq/b)Z; q)_\infty}{(aZ, (aq/b^2)Z; q)_\infty} \right] \\
 &= \mathbb{E} \left[\frac{((aq/b)Z; q)_\infty}{((aq/b^2)Z, aZ; q)_\infty} \right] \\
 &= \sum_{n=0}^1 \sum_{k=0}^\infty \left(\left(\frac{b}{a} \right)^n q^k \left(\left(-\frac{b}{a} \right)^{n-1} q^{k+1}, \left(-\frac{b}{a} \right)^n q^{k+1}, \right. \right. \\
 & \quad \left. \left(\frac{aq}{b} \right) \left(-\frac{b}{a} \right)^n q^k; q \right)_\infty \\
 & \quad \times \left(\left(q, -\frac{aq}{b}, -\frac{b}{a}, \left(\frac{aq}{b^2} \right) \left(-\frac{b}{a} \right)^n q^k, \right. \right. \\
 & \quad \left. \left. a \left(-\frac{b}{a} \right)^n q^k; q \right)_\infty \right)^{-1} \Bigg) \\
 &= \frac{1}{(1-q)(q, -aq/b, -b/a; q)_\infty} \\
 & \quad \times \left((1-q) \sum_{k=0}^\infty \frac{\left(q^{k+1}/(-b/a), q^{k+1}, (aq/b)q^k; q \right)_\infty q^k}{((aq/b^2)q^k, aq^k; q)_\infty} \right. \\
 & \quad \left. - \left(-\frac{b}{a} \right) (1-q) \right. \\
 & \quad \left. \times \sum_{k=0}^\infty \frac{\left(q^{k+1}, (-b/a)q^{k+1}, (aq/b)(-b/a)q^k; q \right)_\infty q^k}{((aq/b^2)(-b/a)q^k, a(-b/a)q^k; q)_\infty} \right) \\
 &= \frac{1}{(1-q)(q, -aq/b, -b/a; q)_\infty} \\
 & \quad \times \int_{-b/a}^1 \frac{(qt/(-b/a), qt, (aq/b)t; q)_\infty d_q t}{((aq/b^2)t, at; q)_\infty} \\
 &= \frac{(1-q)(q, -aq/b, -b/a, aq/b; q)_\infty}{(1-q)(q, -aq/b, -b/a, -q/b, aq/b^2, a; q)_\infty} {}_2\phi_1 \\
 & \quad \times \left(\frac{aq}{b^2}, \frac{q}{b}; q, -b \right) \\
 &= \frac{(aq/b; q)_\infty}{(-q/b, aq/b^2, a; q)_\infty} {}_2\phi_1 \left(\frac{aq}{b^2}, \frac{q}{b}; q, -b \right). \tag{88}
 \end{aligned}$$

Comparing (87) and (88), we have

$$\begin{aligned}
 & {}_2\phi_1 \left(\frac{a, b}{aq}; q, -\frac{q}{b} \right) \\
 &= \frac{(a, -b, aq/b^2; q)_\infty}{(aq/b; q)_\infty} \frac{(aq/b; q)_\infty}{(-q/b, aq/b^2, a; q)_\infty} {}_2\phi_1
 \end{aligned}$$

$$\begin{aligned}
 & \times \left(\frac{aq}{b^2}, \frac{q}{b}; q, -b \right) \\
 &= \frac{(-b; q)_\infty}{(-q/b; q)_\infty} {}_2\phi_1 \left(\frac{aq}{b^2}, \frac{q}{b}; q, -b \right). \tag{89}
 \end{aligned}$$

Using Heine's transformation and q -binomial theorem, we have

$$\begin{aligned}
 & {}_2\phi_1 \left(\frac{a, b}{aq}; q, -\frac{q}{b} \right) \\
 &= \frac{(-b; q)_\infty}{(-q/b; q)_\infty} \frac{(a, -q; q)_\infty}{(aq/b, -b; q)_\infty} {}_2\phi_1 \left(\frac{-\frac{q}{b}, \frac{q}{b}}{-q}; q, a \right) \\
 &= \frac{(a, -q; q)_\infty}{(-q/b, aq/b; q)_\infty} \sum_{n=0}^\infty \frac{(q^2/b^2; q^2)_n}{(q^2; q^2)_n} a^n \tag{90} \\
 &= \frac{(a, -q; q)_\infty}{(-q/b, aq/b; q)_\infty} \frac{(aq^2/b^2; q^2)_\infty}{(a; q^2)_\infty} \\
 &= \frac{(-q; q)_\infty (aq, aq^2/b^2; q^2)_\infty}{(-q/b, aq/b; q)_\infty}.
 \end{aligned}$$

Hence, we obtain (83). \square

Theorem 15 (see [17, page 354, II. 10]). *Bailey's sum formula is*

$${}_2\phi_2 \left(\frac{a, \frac{q}{a}}{-q, b}; q, -b \right) = \frac{(ab, bq/a; q^2)_\infty}{(b; q)_\infty}. \tag{91}$$

Proof. By (19), we have

$$\mathbb{E} \left[\frac{(dX; q)_\infty}{(aX, bX; q)_\infty} \right] = \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} {}_2\phi_2 \left(\frac{a, b}{abx, d}; q, dx \right). \tag{92}$$

Replacing (a, b, d, x) by $(a, q/a, -q, b/q)$ in (92) gives

$$\begin{aligned}
 & P \left(R = \left(\frac{b}{q} \right)^n q^k \right) \\
 &= p_{n,k} \left(\frac{b}{q}; q \right) \\
 &=: \frac{(-b/q)^n q^k ((b/q)^{n-1} q^{k+1}, (b/q)^n q^{k+1}; q)_\infty}{(q, q^2/b, b/q; q)_\infty}, \tag{93}
 \end{aligned}$$

where

$$\begin{aligned}
 & p_{n,k} \left(\frac{b}{q}; q \right) > 0, \quad \sum p_{n,k} \left(\frac{b}{q}; q \right) = 1, \\
 & \frac{b}{q} < 0, \quad 0 < q < 1, \quad n = 0, 1, 2, \dots
 \end{aligned} \tag{94}$$

Hence, we have

$$\begin{aligned} & \mathbb{E} \left[\frac{(-qR; q)_\infty}{(aR, (q/a)R; q)_\infty} \right] \\ &= \frac{(-q, b; q)_\infty}{(a, ab/q, q/a, b/a; q)_\infty} {}_2\phi_2 \left(\begin{matrix} a, \frac{q}{a} \\ b, -q \end{matrix}; q, -b \right). \end{aligned} \quad (95)$$

By using the probability distribution $W(b/q; q)$ and Lemma 2, we calculate the expectation of the random variables $(-qR; q)_\infty/(aR, (q/a)R; q)_\infty$ as follows:

$$\begin{aligned} & \mathbb{E} \left[\frac{(-qR; q)_\infty}{(aR, (q/a)R; q)_\infty} \right] \\ &= \sum_{n=0}^1 \sum_{k=0}^\infty \left(\left(-\frac{b}{q} \right)^n q^k \left(\left(\frac{b}{q} \right)^{n-1} q^{k+1}, \left(\frac{b}{q} \right)^n q^{k+1}, \right. \right. \\ & \quad \left. \left. (-q) \left(\frac{b}{q} \right)^n q^k; q \right)_\infty \right. \\ & \quad \times \left. \left(\left(q, \frac{q^2}{b}, \frac{b}{q}, a \left(\frac{b}{q} \right)^n q^k, \left(\frac{q}{a} \right) \left(\frac{b}{q} \right)^n q^k; q \right)_\infty \right)^{-1} \right) \\ &= \frac{1}{(1-q)(q, q^2/b, b/q; q)_\infty} \\ & \quad \times \left((1-q) \sum_{k=0}^\infty \frac{(q^{k+1}/(b/q), q^{k+1}, (-q)q^k; q)_\infty q^k}{(aq^k, (q/a)q^k; q)_\infty} \right. \\ & \quad \left. - \left(\frac{b}{q} \right) (1-q) \right. \\ & \quad \left. \times \sum_{k=0}^\infty \frac{(q^{k+1}, (b/q)q^{k+1}, (-q)(b/q)q^k; q)_\infty q^k}{(a(b/q)q^k, (q/a)(b/q)q^k; q)_\infty} \right) \\ &= \frac{1}{(1-q)(q, q^2/b, b/q; q)_\infty} \\ & \quad \times \int_{b/q}^1 \frac{(qt/(b/q), qt, -qt; q)_\infty}{(at, (q/a)t; q)_\infty} d_q t \\ &= \frac{1}{(1-q)(q, q^2/b, b/q; q)_\infty} \frac{(1-q)(q, q^2/b, b/q, -q; q)_\infty}{(ab/q, a, q/a; q)_\infty} {}_2\phi_1 \\ & \quad \times \left(\begin{matrix} a, -a \\ -q \end{matrix}; q, \frac{b}{a} \right) \\ &= \frac{(-q; q)_\infty}{(ab/q, a, q/a; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, -a \\ -q \end{matrix}; q, \frac{b}{a} \right). \end{aligned} \quad (96)$$

Comparing (95) and (96), we have

$$\begin{aligned} {}_2\phi_2 \left(\begin{matrix} a, \frac{q}{a} \\ -q, b \end{matrix}; q, -b \right) &= \frac{(b/a; q)_\infty}{(b; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, -a \\ -q \end{matrix}; q, \frac{b}{a} \right) \\ &= \frac{(b/a; q)_\infty}{(b; q)_\infty} \sum_{n=0}^\infty \frac{(a, -a; q)_n}{(q, -q; q)_n} \left(\frac{b}{a} \right)^n \\ &= \frac{(b/a; q)_\infty}{(b; q)_\infty} \sum_{n=0}^\infty \frac{(a^2; q^2)_n}{(q^2; q^2)_n} \left(\frac{b}{a} \right)^n \\ &= \frac{(b/a; q)_\infty}{(b; q)_\infty} \frac{(ab; q^2)_\infty}{(b/a; q^2)_\infty} \\ &= \frac{(ab, bq/a; q^2)_\infty}{(b; q)_\infty}. \end{aligned} \quad (97)$$

Hence, we get (91). \square

Theorem 16 (see [17, page 354, II. 11]). *The Gauss sum formula is*

$${}_2\phi_2 \left(\begin{matrix} a^2, b^2 \\ abq^{1/2}, -abq^{1/2} \end{matrix}; q, -q \right) = \frac{(a^2q, b^2q; q^2)_\infty}{(q, a^2b^2q; q^2)_\infty}. \quad (98)$$

Proof. By (14), we have

$$\begin{aligned} & \mathbb{E} \left[\frac{(dX; q)_\infty}{(aX, bX; q)_\infty} \right] \\ &= \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} {}_2\phi_2 \left(\begin{matrix} a, b \\ abx, d \end{matrix}; q, dx \right). \end{aligned} \quad (99)$$

Replacing (a, b, d, x) by $(a^2, b^2, -abq^{1/2}, q^{1/2}/ab)$ in (99) gives

$$\begin{aligned} P \left(S = \left(\frac{q^{1/2}}{ab} \right)^n q^k \right) \\ = p_{n,k} \left(\frac{q^{1/2}}{ab}; q \right) \\ =: \frac{(-q^{1/2}/ab)^n q^k \left((q^{1/2}/ab)^{n-1} q^{k+1}, (q^{1/2}/ab)^n q^{k+1}; q \right)_\infty}{(q, abq^{1/2}, q^{1/2}/ab; q)_\infty}, \end{aligned} \quad (100)$$

where

$$\begin{aligned} p_{n,k} \left(\frac{q^{1/2}}{ab}; q \right) &> 0, \quad \sum p_{n,k} \left(\frac{q^{1/2}}{ab}; q \right) = 1, \\ \frac{q^{1/2}}{ab} &< 0, \quad 0 < q < 1, \quad n = 0, 1, \quad k = 0, 1, 2, \dots. \end{aligned} \quad (101)$$

Hence,

$$\begin{aligned} \mathbb{E} \left[\frac{(-abq^{1/2}S; q)_\infty}{(a^2S, b^2S; q)_\infty} \right] &= \frac{(-abq^{1/2}, abq^{1/2}; q)_\infty}{(a^2, aq^{1/2}/b, b^2, bq^{1/2}/a; q)_\infty} {}_2\phi_2 \\ &\quad \times \left(\begin{matrix} a^2, b^2 \\ abq^{1/2}, -abq^{1/2} \end{matrix}; q, -q \right). \end{aligned} \quad (102)$$

By using the probability distribution $W(q^{1/2}/ab; q)$ and employing Andrews-Askey q -integral (13) of Lemma 2, we calculate the expectation of the random variables $(-abq^{1/2}S; q)_\infty/(a^2S, b^2S; q)_\infty$ as follows:

$$\begin{aligned} \mathbb{E} \left[\frac{(-abq^{1/2}S; q)_\infty}{(a^2S, b^2S; q)_\infty} \right] &= \sum_{n=0}^1 \sum_{k=0}^\infty \left(\left(-\frac{q^{1/2}}{ab} \right)^n q^k \left(\left(\frac{q^{1/2}}{ab} \right)^{n-1} q^{k+1}, \left(\frac{q^{1/2}}{ab} \right)^n q^{k+1}, \right. \right. \\ &\quad \left. \left. (-abq^{1/2}) \left(\frac{q^{1/2}}{ab} \right)^n q^k; q \right)_\infty \right. \\ &\quad \times \left(\left(q, abq^{1/2}, \frac{q^{1/2}}{ab}, a^2 \left(\frac{q^{1/2}}{ab} \right)^n q^k, \right. \right. \\ &\quad \left. \left. b^2 \left(\frac{q^{1/2}}{ab} \right)^n q^k; q \right)_\infty \right)^{-1} \Bigg) \\ &= \frac{1}{(1-q)(q, abq^{1/2}, q^{1/2}/ab; q)_\infty} \\ &\quad \times \left((1-q) \sum_{k=0}^\infty \frac{(q^{k+1}/(q^{1/2}/ab), q^{k+1}, (-abq^{1/2})q^k; q)_\infty q^k}{(a^2q^k, b^2q^k; q)_\infty} \right. \\ &\quad \left. - \left(\frac{q^{1/2}}{ab} \right) (1-q) \right. \\ &\quad \left. \times \sum_{k=0}^\infty \frac{(q^{k+1}, (q^{1/2}/ab)q^{k+1}, (-abq^{1/2})(q^{1/2}/ab)q^k; q)_\infty q^k}{(a^2(q^{1/2}/ab)q^k, b^2(q^{1/2}/ab)q^k; q)_\infty} \right) \\ &= \frac{1}{(1-q)(q, abq^{1/2}, q^{1/2}/ab; q)_\infty} \\ &\quad \times \int_{q^{1/2}/ab}^1 \frac{(qt/(q^{1/2}/ab), qt, -abq^{1/2}t; q)_\infty}{(a^2t, b^2t; q)_\infty} d_q t \end{aligned}$$

$$\begin{aligned} &= \frac{(1-q)(q, abq^{1/2}, q^{1/2}/ab, -abq^{1/2}; q)_\infty}{(1-q)(q, abq^{1/2}, q^{1/2}/ab, aq^{1/2}/b, a^2, b^2; q)_\infty} {}_2\phi_1 \\ &\quad \times \left(\begin{matrix} a^2, -\frac{aq^{1/2}}{b} \\ -abq^{1/2} \end{matrix}; q, \frac{bq^{1/2}}{a} \right) \\ &= \frac{(-abq^{1/2}; q)_\infty}{(aq^{1/2}/b, a^2, b^2; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a^2, -\frac{aq^{1/2}}{b} \\ -abq^{1/2} \end{matrix}; q, \frac{bq^{1/2}}{a} \right). \end{aligned} \quad (103)$$

Comparing (102) and (103), we have

$$\begin{aligned} &{}_2\phi_2 \left(\begin{matrix} a^2, b^2 \\ abq^{1/2}, -abq^{1/2} \end{matrix}; q, -q \right) \\ &= \frac{(a^2, aq^{1/2}/b, b^2, bq^{1/2}/a; q)_\infty}{(-abq^{1/2}, abq^{1/2}; q)_\infty} \frac{(-abq^{1/2}; q)_\infty}{(aq^{1/2}/b, a^2, b^2; q)_\infty} {}_2\phi_1 \\ &\quad \times \left(\begin{matrix} a^2, -\frac{aq^{1/2}}{b} \\ -abq^{1/2} \end{matrix}; q, \frac{bq^{1/2}}{a} \right) \\ &= \frac{(bq^{1/2}/a; q)_\infty}{(abq^{1/2}; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a^2, -\frac{aq^{1/2}}{b} \\ -abq^{1/2} \end{matrix}; q, \frac{bq^{1/2}}{a} \right). \end{aligned} \quad (104)$$

Using Heine's transformation formula, we have

$$\begin{aligned} &{}_2\phi_1 \left(\begin{matrix} a^2, -\frac{aq^{1/2}}{b} \\ -abq^{1/2} \end{matrix}; q, \frac{bq^{1/2}}{a} \right) \\ &= \frac{(b^2, -q; q)_\infty}{(-abq^{1/2}, bq^{1/2}/a; q)_\infty} {}_2\phi_1 \left(\begin{matrix} aq^{1/2}, -\frac{aq^{1/2}}{b} \\ -q \end{matrix}; q, b^2 \right). \end{aligned} \quad (105)$$

Substituting (105) into (104) yields

$$\begin{aligned} &{}_2\phi_2 \left(\begin{matrix} a^2, b^2 \\ abq^{1/2}, -abq^{1/2} \end{matrix}; q, -q \right) \\ &= \frac{(bq^{1/2}/a; q)_\infty}{(abq^{1/2}; q)_\infty} \frac{(b^2, -q; q)_\infty}{(-abq^{1/2}, bq^{1/2}/a; q)_\infty} {}_2\phi_1 \\ &\quad \times \left(\begin{matrix} aq^{1/2}, -\frac{aq^{1/2}}{b} \\ -q \end{matrix}; q, b^2 \right) \\ &= \frac{(b^2, -q; q)_\infty}{(abq^{1/2}, -abq^{1/2}; q)_\infty} \sum_{n=0}^\infty \frac{(aq^{1/2}/b, -aq^{1/2}/b; q)_n}{(q, -q; q)_n} (b^2)^n \end{aligned}$$

$$\begin{aligned}
&= \frac{(b^2, -q; q)_\infty}{(abq^{1/2}, -abq^{1/2}; q)_\infty} \sum_{n=0}^{\infty} \frac{(a^2 q/b^2; q^2)_n}{(q^2; q^2)_n} (b^2)^n \\
&= \frac{(b^2, -q; q)_\infty}{(abq^{1/2}, -abq^{1/2}; q)_\infty} \frac{(a^2 q; q^2)_\infty}{(b^2; q^2)_\infty}.
\end{aligned} \tag{106}$$

Noting that

$$\begin{aligned}
(a^2; q^2)_\infty &= (a; q)_\infty (-a; q)_\infty, \\
(aq; q^2)_\infty &= \frac{(a; q)_\infty}{(a; q^2)_\infty},
\end{aligned} \tag{107}$$

we have

$$\begin{aligned}
(-q; q)_\infty &= \frac{(-q; q)_\infty (q; q)_\infty}{(q; q)_\infty} = \frac{(q^2; q^2)_\infty}{(q; q)_\infty} = \frac{1}{(q; q^2)_\infty}, \\
\frac{(b^2; q)_\infty}{(b^2; q^2)_\infty} &= (b^2 q; q^2)_\infty, \\
(abq^{1/2}; q)_\infty (-abq^{1/2}; q)_\infty &= (a^2 b^2 q; q^2)_\infty.
\end{aligned} \tag{108}$$

Substituting (108) into (106) yields

$${}_2\phi_2 \left(\begin{matrix} a^2, b^2 \\ abq^{1/2}, -abq^{1/2} \end{matrix}; q, -q \right) = \frac{(a^2 q, b^2 q; q^2)_\infty}{(q, a^2 b^2 q; q^2)_\infty}. \tag{109}$$

Hence, we obtain (98). \square

Theorem 17 (see [17, page 354, II. 5]). *A sum formula of ${}_1\phi_1$ is*

$${}_1\phi_1 \left(\begin{matrix} a \\ c \end{matrix}; q, \frac{c}{a} \right) = \frac{(c/a; q)_\infty}{(c; q)_\infty}. \tag{110}$$

Proof. Letting $d = a = b$ in (19) of Theorem 4, we obtain

$$\mathbb{E} \left[\frac{1}{(ax; q)_\infty} \right] = \frac{(a^2 x; q)_\infty}{(ax, ax, a; q)_\infty} {}_1\phi_1 \left(\begin{matrix} a \\ a^2 x \end{matrix}; q, ax \right). \tag{111}$$

Letting $d = a = c$ and $b = 0$ in (14) of Theorem 3 or setting $d = c$ and $a = 0$ and replacing b by a in (14) of Theorem 3, we obtain

$$\mathbb{E} \left[\frac{1}{(ax; q)_\infty} \right] = \frac{1}{(ax, a; q)_\infty}. \tag{112}$$

Comparing (111) and (112) gives

$$\frac{(a^2 x; q)_\infty}{(ax, ax, a; q)_\infty} {}_1\phi_1 \left(\begin{matrix} a \\ a^2 x \end{matrix}; q, ax \right) = \frac{1}{(ax, a; q)_\infty}; \tag{113}$$

that is,

$${}_1\phi_1 \left(\begin{matrix} a \\ a^2 x \end{matrix}; q, ax \right) = \frac{(ax; q)_\infty}{(a^2 x; q)_\infty}. \tag{114}$$

Replacing $(a, a^2 x)$ by (a, c) , we get

$${}_1\phi_1 \left(\begin{matrix} a \\ c \end{matrix}; q, \frac{c}{a} \right) = \frac{(c/a; q)_\infty}{(c; q)_\infty}. \tag{115}$$

This proof is complete. \square

Theorem 18 (see [17, page 354, II. 1, II. 2]). *The two q -exponential functions are*

$$\begin{aligned}
e_q(z) &= \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty} \quad \text{for } |z| < 1, \\
E_q(z) &= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{(q; q)_n} = (-z; q)_\infty.
\end{aligned} \tag{116}$$

Proof. Setting $a = b = d = 0$ and then replacing c by a in (14), we obtain

$$\mathbb{E} \left[\frac{1}{(ax; q)_\infty} \right] = \frac{1}{(a; q)_\infty} {}_1\phi_0 \left(\begin{matrix} 0 \\ - \end{matrix}; q, ax \right). \tag{117}$$

Letting $d = a = c$ and $b = 0$ in (14) or letting $d = c$ and $a = 0$ and replacing b by a in (14), we obtain

$$\mathbb{E} \left[\frac{1}{(ax; q)_\infty} \right] = \frac{1}{(ax, a; q)_\infty}. \tag{118}$$

Comparing (117) and (118) gives

$$\frac{1}{(a; q)_\infty} {}_1\phi_0 \left(\begin{matrix} 0 \\ - \end{matrix}; q, ax \right) = \frac{1}{(ax, a; q)_\infty}. \tag{119}$$

From the above formula, we obtain

$${}_1\phi_0 \left(\begin{matrix} 0 \\ - \end{matrix}; q, ax \right) = \frac{1}{(ax; q)_\infty}. \tag{120}$$

Replacing ax by z gives

$${}_1\phi_0 \left(\begin{matrix} 0 \\ - \end{matrix}; q, z \right) = \frac{1}{(z; q)_\infty}; \tag{121}$$

that is,

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty} \quad \text{for } |z| < 1. \tag{122}$$

Similarly, setting $d = a$ and $b = 0$ in (19) of Theorem 4, we obtain

$$\mathbb{E}[1] = \frac{1}{(ax; q)_\infty} {}_0\phi_1 \left(\begin{matrix} - \\ - \end{matrix}; q, ax \right). \tag{123}$$

And obviously letting $d = c$ and $a = b = 0$ in (14) of Theorem 3, we obtain

$$\mathbb{E}[1] = 1. \tag{124}$$

Comparing (123) and (124), we obtain

$$\frac{1}{(ax;q)_\infty} {}_0\phi_0 \left(\begin{matrix} - \\ - \end{matrix}; q, ax \right) = 1. \quad (125)$$

Replacing ax by $-z$, we get

$${}_0\phi_0 \left(\begin{matrix} - \\ - \end{matrix}; q, -z \right) = (-z; q)_\infty; \quad (126)$$

that is,

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{(q; q)_n} = (-z; q)_\infty. \quad (127)$$

□

Remark 19. In the present paper we obtain the part transformation and sum formulas of the q -series by applying the probabilistic method. We hope to find and construct another probability distribution in order to prove the transformation and sum formulas of the bilateral basic hypergeometric series, for example, Ramanujan and Bailey sum formulas and so forth.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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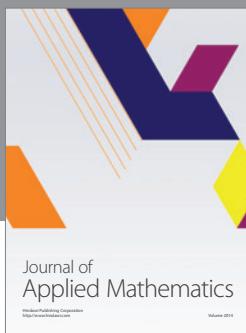
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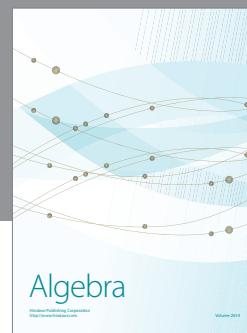
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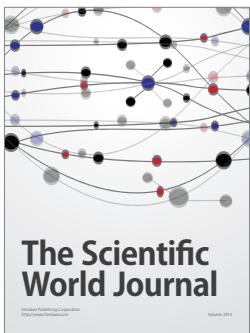
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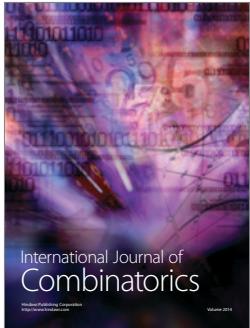


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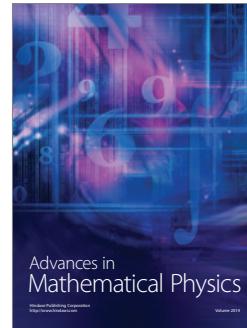
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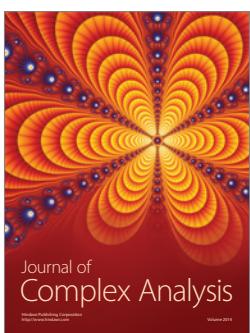
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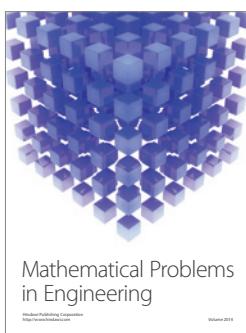
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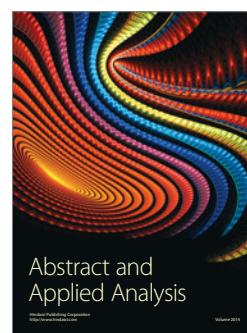
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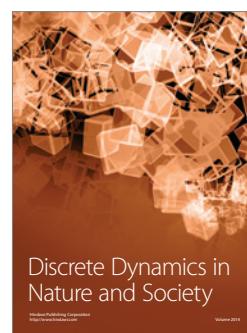
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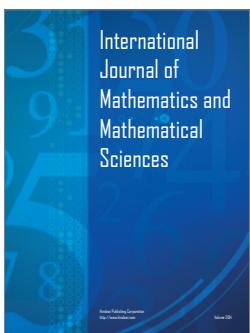
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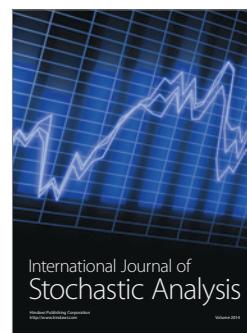
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