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Research Article

On Some Transverse Geometrical Structures of Lifted Foliation to Its Conormal Bundle

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We consider the lift of a foliation to its conormal bundle and some transverse geometrical structures associated with this foliation are studied. We introduce a good vertical connection on the conormal bundle and, moreover, if the conormal bundle is endowed with a transversal Cartan metric, we obtain that the lifted foliation to its conormal bundle is a Riemannian one. Also, some transversally framed $f(3,\varepsilon)$-structures of corank 2 on the normal bundle of lifted foliation to its conormal bundle are introduced and an almost (para)contact structure on a transverse Liouville distribution is obtained.

1. Introduction and Preliminaries

The study of the lift of transversal Finsler foliations to their normal bundle using the technique of good vertical connection was initiated by Miernowski and Mozgawa [1] where it is proved that the lifted foliation is a Riemannian one. Also, using different methods, some connections between foliations and Lagrangians (or Hamiltonians) in order to recover Riemannian foliations are investigated in the recent papers [2–5]. Our aim in this paper is to extend the study from [1] for the case of lifted foliation to its conormal bundle. In this sense we introduce a good vertical connection on the conormal bundle and we give an application of it in order to obtain that the lifted foliation is a Riemannian one in the case when the conormal bundle is endowed with a transversal Cartan metric. Moreover, in this case, some transversally framed $f(3,\varepsilon)$-structures and an almost (para)contact structure associated with lifted foliation are investigated.

The methods used here are similarly and closely related to those used in [1, 6] for the case of transversal Finsler foliations.

Let us consider M an $(n+m)$-dimensional manifold which will be assumed to be connected and orientable.

Definition 1. A codimension n foliation $\mathcal{F}$ on M is defined by a foliated cocycle $\{U_i,\varphi_i,f_{i,j}\}$ such that

(i) $\{U_i\}$, $i\in I$, is an open covering of M;

(ii) for every $i\in I$, $\varphi_i:U_i\rightarrow N$ are submersions, where N is an n-dimensional manifold, called transversal manifold;

(iii) the maps $f_{i,j}:\varphi_i(U_i\cap U_j)\rightarrow\varphi_j(U_i\cap U_j)$ satisfy

$$\varphi_j=f_{i,j}\circ\varphi_i \tag{1}$$

for every $(i,j)\in I\times I$ such that $U_i\cap U_j\neq\emptyset$.

Every fibre of φ_i is called a plaque of the foliation. Condition (1) says that on the intersection $U_i\cap U_j$ the plaques defined, respectively, by φ_i and φ_j coincide. The manifold M is decomposed into a family of disjoint immersed connected submanifolds of dimension m; each of these submanifolds is called a leaf of $\mathcal{F}$.

By $T\mathcal{F}$ we denote the tangent bundle to $\mathcal{F}$ and $\Gamma(\mathcal{F})$ is the space of its global sections, that is, vector fields tangent to $\mathcal{F}$, and by $Q\mathcal{F}=TM/T\mathcal{F}$ we denote the normal bundle of $\mathcal{F}$.

In this paper, a system of local coordinates adapted to the foliation $\mathcal{F}$ means coordinates $(x^1,\ldots,x^n,y^1,\ldots,y^m)$ on an open subset U on which the foliation is trivial and defined by the equations $dx^a=0$, $a=1,\ldots,n$.

We notice that the total spaces of the conormal bundle $Q^*\mathcal{F}$ of $\mathcal{F}$ carry a natural foliation $\widetilde{\mathcal{F}}$ of codimension $2n$ such that the leaves of $\widetilde{\mathcal{F}}$ are covering spaces of the leaves of $\mathcal{F}$, and it is called the natural lift of $\mathcal{F}$ to its conormal bundle $Q^*\mathcal{F}$.

If we denote by $\{dx^a\}$, $a = 1,\ldots,n$, the corresponding local coframe on $Q^*\mathcal{F}$, then we can induce a chart (x^a, p_a, y^u) on $Q^*\mathcal{F}$ where $p = p_a dx^a \in \Gamma(Q^*\mathcal{F})$, and the system of equations $x^a = \text{const.}$, $p_a = \text{const.}$ defines the foliation $\widetilde{\mathcal{F}}$.

Let $Q\widetilde{\mathcal{F}} = T(Q^*\mathcal{F})/T\widetilde{\mathcal{F}}$ be the normal bundle of the foliated manifold $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$. The vectors $\{\partial/\partial x^a, \partial/\partial p_a\}$, $a = 1,\ldots,n$, form a natural frame of $Q\widetilde{\mathcal{F}}$ at the point $(x^a, p_a, y^u) \in Q^*\mathcal{F}$. The canonical projection $\pi : Q^*\mathcal{F} \to M$ given by $\pi(x^a, p_a, y^u) = (x^a, y^u)$ induces another projection $\pi_* : T(Q^*\mathcal{F}) \to TM$ which maps the vectors tangent to $\widetilde{\mathcal{F}}$ in the vectors tangent to $\mathcal{F}$. Thus, π_* induces a mapping $\tilde{\pi}_* : Q\widetilde{\mathcal{F}} \to Q\mathcal{F}$ and is denoted by $V(Q^*\mathcal{F}) = \ker\tilde{\pi}_*$ which is a vertical bundle spanned by the vectors $\{\partial/\partial p_a\}$, $a = 1,\ldots,n$.

Lemma 2. *Let $o : M \to Q^*\mathcal{F}$ be the zero section of the conormal bundle $Q^*\mathcal{F}$. Then the set $o(M)$ is saturated on $Q^*\mathcal{F}$ with foliation $\widetilde{\mathcal{F}}$.*

2. Good Vertical Connection on $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$

The purpose of this section is to define a linear connection $\nabla : \mathcal{X}(V(Q^*\mathcal{F})) \to \mathcal{X}(T^*(\widetilde{Q}^*\mathcal{F}) \otimes V(Q^*\mathcal{F}))$ related to considered foliated structure, where $\widetilde{Q}^*\mathcal{F} = Q^*\mathcal{F} - o(M)$. Since we have the foliated manifold $(\widetilde{Q}^*\mathcal{F}, \widetilde{\mathcal{F}})$, we are looking for a Bott connection such that for any vector field X tangent to $\widetilde{\mathcal{F}}$ and any transversal vector field Y we have

$$\nabla_X Y = p_{Q\widetilde{\mathcal{F}}}\left(\left[X, \widetilde{Y}\right]\right), \quad (2)$$

where $p_{Q\widetilde{\mathcal{F}}} : T(Q^*\mathcal{F}) \to Q\widetilde{\mathcal{F}}$ is the canonical projection and $p_{Q\widetilde{\mathcal{F}}}(\widetilde{Y}) = Y$.

Let us consider now the $\widetilde{\mathcal{F}}$-transversal Hamilton-Liouville vector field defined by $C^* : Q^*\mathcal{F} \to V(Q^*\mathcal{F})$, $C^*(x^a, p_a, y^u) = p_a(\partial/\partial p_a)$. It can be checked that this definition is well posed. From the definition of the Bott connection, the following lemma holds.

Lemma 3. *Let $\nabla : \mathcal{X}(V(Q^*\mathcal{F})) \to \mathcal{X}(T^*(\widetilde{Q}^*\mathcal{F}) \otimes V(Q^*\mathcal{F}))$ be a Bott connection. Then $\nabla_X C^* = 0$ for every vector field tangent to $\widetilde{\mathcal{F}}$.*

Now, consider the local frame $\{\partial/\partial x^a, \partial/\partial p_a, \partial/\partial y^u\}$ of $T(Q^*\mathcal{F})$ and recall that the vectors $\{\partial/\partial p_a\}$ form the basis of $V(Q^*\mathcal{F})$. With these settings we put

$$\nabla_{\partial/\partial x^a}\frac{\partial}{\partial p_b} = \Gamma^b_{ac}\frac{\partial}{\partial p_c}, \qquad \nabla_{\partial/\partial p_a}\frac{\partial}{\partial p_b} = \Gamma^{ab}_c\frac{\partial}{\partial p_c},$$
$$\nabla_{\partial/\partial y^u}\frac{\partial}{\partial p_b} = \Gamma^b_{uc}\frac{\partial}{\partial p_c}. \quad (3)$$

From the above formulas it follows that

$$\Gamma^b_{uc} = 0, \qquad \nabla_{\partial/\partial p_a} C^* = \left(\delta^a_c + p_b\Gamma^{ab}_c\right)\frac{\partial}{\partial p_c}. \quad (4)$$

The Bott connection ∇ allows us to define a mapping

$$L : \mathcal{X}\left(Q\widetilde{\mathcal{F}}\right) \longrightarrow \mathcal{X}\left(V\left(Q^*\mathcal{F}\right)\right), \qquad L(X) = \nabla_{\widetilde{X}} C^*, \quad (5)$$

where $p_{Q\widetilde{\mathcal{F}}}(\widetilde{X}) = X$. If we denote by Λ the restriction of the linear mapping L to the bundle $V(Q^*\mathcal{F})$, then we can state the following.

Definition 4. The Bott connection ∇ is said to be a good vertical connection if $\Lambda : V(Q^*\mathcal{F}) \to V(Q^*\mathcal{F})$ is a bundle isomorphism.

Observe that ∇ is a good vertical connection if and only if the matrix $\delta^a_c + p_b\Gamma^{ab}_c$ is nondegenerated. If we put $H(Q^*\mathcal{F}) = \ker L$, then we can split the bundle $Q\widetilde{\mathcal{F}}$ into direct sum:

$$Q\widetilde{\mathcal{F}} = H\left(Q^*\mathcal{F}\right) \oplus V\left(Q^*\mathcal{F}\right). \quad (6)$$

The coefficients of the mapping L in the basis $\{\partial/\partial x^a, \partial/\partial p_a\}$ of $Q\widetilde{\mathcal{F}}$ are

$$L\left(\frac{\partial}{\partial x^a}\right) = p_b\Gamma^b_{ac}\frac{\partial}{\partial p_c},$$
$$L\left(\frac{\partial}{\partial p_a}\right) = \left(\delta^a_c + p_b\Gamma^{ab}_c\right)\frac{\partial}{\partial p_c} = L^a_c\frac{\partial}{\partial p_c}. \quad (7)$$

It is easy to check that the vectors $\delta/\delta x^a = \partial/\partial x^a + N_{ab}(\partial/\partial p_b)$, where $N_{ab} = -(L^{-1})^c_b p_d\Gamma^d_{ac}$, form a basis of $\ker L$. In the sequel we will use the basis $\{\delta/\delta x^a, \partial/\partial p_a\}$, called adapted, as well as its dual $\{dx^a, \delta p_a = dp_a - N_{ab}dx^b\}$. Using this coframe we can define the local connection forms by

$$\nabla\frac{\partial}{\partial p_b} = \omega^b_a \otimes \frac{\partial}{\partial p_a}, \quad (8)$$

where

$$\omega^b_a = \Gamma^b_{ca}dx^c + \Gamma^{bc}_a dp_c = \left(\Gamma^b_{ca} + \Gamma^{bd}_a N_{dc}\right)dx^c + \Gamma^{bc}_a\delta p_c$$
$$= H^b_{ca}dx^c + \Gamma^{bc}_a\delta p_c. \quad (9)$$

Notice that $H^b_{ca}p_b = N_{ca}$. The formula $\theta(\partial/\partial p_a) = \delta/\delta x^a$ defines a linear mapping $\theta : V(Q^*\mathcal{F}) \to H(Q^*\mathcal{F})$. This mapping allows us to extend the connection ∇ to the horizontal bundle $H(Q^*\mathcal{F})$ by

$$\nabla_X Y = \theta\left(\nabla_X\theta^{-1}(Y)\right), \quad (10)$$

where $Y \in \Gamma(H(Q^*\mathcal{F}))$, $X \in \Gamma(T(Q^*\mathcal{F}))$. In this way we construct a linear connection in $Q\widetilde{\mathcal{F}}$:

$$\nabla_X Y = \nabla_X(\nu(Y)) + \nabla_X(Y - \nu(Y)), \quad (11)$$

where $Y \in \Gamma(Q\widetilde{\mathcal{F}})$, $X \in \Gamma(T(Q^*\mathcal{F}))$ and $\nu : Q\widetilde{\mathcal{F}} \to V(Q^*\mathcal{F})$ is the vertical projection from decomposition (6). In particular we have

$$\nabla\frac{\delta}{\delta x^a} = \omega^b_a \otimes \frac{\delta}{\delta x^b}, \quad (12)$$

where ω^b_a is given in (9).

If $\varphi \in \Gamma(Q^*\widetilde{\mathcal{F}} \otimes Q\widetilde{\mathcal{F}})$ is a 1-form with values in $Q\widetilde{\mathcal{F}}$, locally given by

$$\varphi = \varphi^a \otimes \frac{\delta}{\delta x^a} + \varphi_b \otimes \frac{\partial}{\partial p_b}, \quad (13)$$

then, following [1, 7], we can define an exterior differential $D\varphi$ by putting

$$D\varphi = \left(d\varphi^a - \varphi^a \wedge \omega_a^c\right) \otimes \frac{\delta}{\delta x^a} + \left(d\varphi_b - \varphi_b \wedge \omega_c^b\right) \otimes \frac{\partial}{\partial p_c}. \quad (14)$$

A straightforward calculus shows that the above formula is well defined.

The bundle $Q^*\widetilde{\mathcal{F}} \otimes Q\widetilde{\mathcal{F}}$ admits a natural section η given by

$$\eta = dx^a \otimes \frac{\partial}{\partial x^a} + dp_b \otimes \frac{\partial}{\partial p_b} = dx^a \otimes \frac{\delta}{\delta x^a} + \delta p_b \otimes \frac{\partial}{\partial p_b}. \quad (15)$$

It is clear that the form η is well defined.

Definition 5. The form $\zeta = D\eta$ is called the torsion form of the connection ∇.

Locally the form ζ can be expressed as follows:

$$D\eta = \left(-dx^a \wedge \omega_a^c\right) \otimes \frac{\delta}{\delta x^c} + \left(d\left(\delta p_b\right) - \delta p_c \wedge \omega_b^c\right) \otimes \frac{\partial}{\partial p_b} = \zeta^c \otimes \frac{\delta}{\delta x^c} + \zeta_b \otimes \frac{\partial}{\partial p_b}, \quad (16)$$

where

$$\zeta^c = \frac{1}{2}\left(H_{ea}^c - H_{ae}^c\right) dx^a \wedge dx^e - \Gamma_a^{ce} dx^a \wedge \delta p_e,$$
$$\zeta_b = -dN_{cb} \wedge dx^c - H_{eb}^a \delta p_a \wedge dx^e - \Gamma_b^{ae} \delta p_a \wedge \delta p_e. \quad (17)$$

3. Transversal Cartan Metrics on $Q^*\mathcal{F}$ and Riemannian Foliations

As in the case of transversal Finsler metrics on the normal bundle of a foliation, [1, 3], a transversal Cartan metric on $Q^*\mathcal{F}$ is a basic function (with respect to the lifted foliation $\widetilde{\mathcal{F}}$) $K : Q^*\mathcal{F} \to [0, \infty)$ which has the following properties:

(i) K is C^∞ on $\widetilde{Q}^*\mathcal{F}$;

(ii) $K(x, \lambda p) = \lambda K(x, p)$ for all $\lambda > 0$;

(iii) the $n \times n$ matrix (g^{ab}), where $g^{ab} = (1/2)(\partial^2 K^2/\partial p_a \partial p_a)$, is positive definite at all points of $\widetilde{Q}^*\mathcal{F}$.

Also $K(x, p) > 0$, whenever $p \neq 0$. As usual, [8], the properties of K imply that

$$p^a = g^{ab} p_b, \qquad p_a = g_{ab} p^b, \qquad K^2 = g^{ab} p_a p_b = p_a p^a,$$
$$C^{abc} p_c = C^{acb} p_c = C^{cab} p_c = 0, \quad (18)$$

where (g_{ab}) is the inverse matrix of (g^{ba}) and we have put $p^a = (1/2)(\partial K^2/\partial p_a)$, $C^{abc} = -(1/4)(\partial^3 K^2/\partial p_a \partial p_b \partial p_c)$.

Also, g^{ab} determines a metric structure on $V(Q^*\mathcal{F})$ by setting

$$G^v(X, Y) = g^{ab}(x, p) X_a(x, y, p) Y_b(x, y, p), \quad (19)$$

for every $X = X_a(x, y, p)(\partial/\partial p_a)$ and $Y = Y_b(x, y, p)(\partial/\partial p_b) \in \Gamma(V(Q^*\mathcal{F}))$.

Similar reasons as for transversal Finsler foliations (see Theorem 3.1 from [1]) lead to the following result.

Theorem 6. *Let $K : Q^*\mathcal{F} \to [0, \infty)$ be a transversal Cartan metric and let G^v be the Riemannian metric on $V(Q^*\mathcal{F})$ induced by K as in (19). Then there exists exactly one Bott vertical connection $\nabla : \mathcal{X}(V(Q^*\mathcal{F})) \to \mathcal{X}(T^*(\widetilde{Q}^*\mathcal{F}) \otimes V(Q^*\mathcal{F}))$ such that*

(i) *∇ is a good vertical connection;*

(ii) *if $X, Y \in \Gamma(V(Q^*\mathcal{F}))$ and $Z \in \Gamma(T(\widetilde{Q}^*\mathcal{F}))$, then*

$$ZG^v(X, Y) = G^v(\nabla_Z X, Y) + G^v(X, \nabla_Z Y); \quad (20)$$

(iii) *$\zeta(X, Y) = 0$ for every $X, Y \in \Gamma(V(Q^*\mathcal{F}))$;*

(iv) *$\zeta(X, Y) \in \Gamma(V(Q^*\mathcal{F}))$ for every $X, Y \in \Gamma(H(Q^*\mathcal{F}))$.*

Also, the isomorphism θ does not depend on the coordinates along the leaves of $\widetilde{\mathcal{F}}$; so the Riemannian metric in $Q\widetilde{\mathcal{F}}$ defined by $G = G^h + G^v$, where $G^h(X, Y) = G^v(\theta^{-1}(X), \theta^{-1}(Y))$ for every $X, Y \in \Gamma(H(Q^*\mathcal{F}))$ and $G(X, Y) = 0$ for every $X \in \Gamma(H(Q^*\mathcal{F}))$ and $Y \in \Gamma(V(Q^*\mathcal{F}))$, is a transversal Riemannian metric for the lifted foliation $\widetilde{\mathcal{F}}$ to the conormal bundle $Q^*\mathcal{F}$ of $\mathcal{F}$. Hence, we can consider the following.

Theorem 7. *If the conormal bundle of foliation $\mathcal{F}$ is endowed with a transversal Cartan metric, then the lifted foliation $\widetilde{\mathcal{F}}$ to the conormal bundle $Q^*\mathcal{F}$ is Riemannian.*

4. Transversally Framed $f(3,\varepsilon)$-Structures on $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$

The study of structures on manifolds defined by a tensor field satisfying $f^3 \pm f = 0$ has the origin in a paper by Yano [9]. Later on, these structures have been generically called f-structures. On the tangent manifold of a Finsler space, the notion of framed $f(3, 1)$-structure was defined and studied by Anastasiei in [10] and on the cotangent bundle of a Cartan space the study is continued in [11, 12]. Taking into account that the conormal bundle $Q^*\mathcal{F}$ has a local model of a cotangent manifold, in this section we extend the study concerning f-structures in our context.

Let $\varepsilon = \pm 1$. A framed $f(3, \varepsilon)$-structure of corank s on a $(2n + s)$-dimensional manifold N is a natural generalization of an almost contact structure on N (for $\varepsilon = 1$) and of an almost paracontact structure on N (for $\varepsilon = -1$), respectively, and it is a triplet $(f, (\xi_i), (\omega^i))$, $i = 1, \ldots, s$, where f is a tensor field of type $(1,1)$, (ξ_i) are vector fields, and (ω^i) are 1-forms on N such that

$$\omega^i(\xi_j) = \delta^i_j, \qquad f(\xi_i) = 0, \qquad \omega^i \circ f = 0,$$
$$f^2 = -\varepsilon\left(I - \sum_i \omega^i \otimes \xi_i\right), \tag{21}$$

where I denotes the Kronecker tensor field on N. The name of $f(3,\varepsilon)$-structure was suggested by the identity $f^3 + \varepsilon f = 0$. For an account of such kind of structures, we refer to [13].

Let us consider now that $Q^*\mathcal{F}$ is endowed with a transversal Cartan metric K. The linear operator ϕ given in the local adapted basis of $Q\widetilde{\mathcal{F}}$ by

$$\phi\left(\frac{\delta}{\delta x^a}\right) = -\varepsilon g_{ab}\frac{\partial}{\partial p_b}, \qquad \phi\left(\frac{\partial}{\partial p_a}\right) = g^{ab}\frac{\delta}{\delta x^b} \tag{22}$$

defines an almost complex structure on $Q\widetilde{\mathcal{F}}$ for $\varepsilon = 1$ and an almost paracomplex structure on $Q\widetilde{\mathcal{F}}$ for $\varepsilon = -1$, respectively. We also have

$$G(\phi(X), \phi(Y)) = G(X,Y), \quad \forall X, Y \in \Gamma(Q\widetilde{\mathcal{F}}). \tag{23}$$

Let us put $\xi_1 = (p^a/K)(\delta/\delta x^a)$ and $\xi_2 = (p_a/K)(\partial/\partial p_a) = (1/K)C^*$. Thus, we have two global transverse vector fields on $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$ which are linearly independent. The first is transversally horizontal and the second one is transversally vertical.

From the definition of ϕ it follows

$$\phi(\xi_1) = -\varepsilon\xi_2, \qquad \phi(\xi_2) = \xi_1. \tag{24}$$

Now, if we consider the dual transverse 1-forms of ξ_1 and ξ_2, respectively, locally given by $\omega^1 = (p_a/K)dx^a$ and $\omega^2 = (p^a/K)\delta p_a$, then we easily check that

$$\omega^1 \circ \phi = \omega^2, \qquad \omega^2 \circ \phi = -\varepsilon\omega^1,$$
$$\omega^1(X) = G(X, \xi_1), \quad \omega^2(X) = G(X, \xi_2), \quad \forall X \in \Gamma(Q\widetilde{\mathcal{F}}). \tag{25}$$

Next, using ϕ, ξ_i, and ω^i, $i \in \{1,2\}$, we construct the transverse tensor field f of type $(1,1)$ on $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$ by putting

$$f(X) = \phi(X) - \omega^2(X)\xi_1 + \varepsilon\omega^1(X)\xi_2, \quad X \in \Gamma(Q\widetilde{\mathcal{F}}). \tag{26}$$

Using a similar argument as in [6, 10–12] by direct calculus, we obtain the following.

Theorem 8. *The triple $(f, (\xi_i), (\omega^i))$, $i \in \{1,2\}$, provides some transversally framed $f(3,\varepsilon)$-structures of corank 2 on $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$; that is, the following hold:*

(i) $\omega^i(\xi_j) = \delta^i_j$, $f(\xi_i) = 0$, $\omega^i \circ f = 0$;

(ii) $f^2 = -\varepsilon(I - \omega^1 \otimes \xi_1 - \omega^2 \otimes \xi_2)$;

(iii) *f is of rank $2n-2$ and $f^3 + \varepsilon f = 0$.*

Theorem 9. *The transversal Riemannian metric G on $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$ satisfies*

$$G(f(X), f(Y)) = G(X,Y) - \omega^1(X)\omega^1(Y) - \omega^2(X)\omega^2(Y), \quad X, Y \in \Gamma(Q\widetilde{\mathcal{F}}). \tag{27}$$

For $\varepsilon = 1$ we put

$$\Phi(X,Y) = G(f(X), Y), \quad X, Y \in \Gamma(Q\widetilde{\mathcal{F}}). \tag{28}$$

By using Theorems 8 and 9, we obtain

$$\Phi(X,Y) = -\Phi(Y,X), \quad X, Y \in \Gamma(Q\widetilde{\mathcal{F}}). \tag{29}$$

Thus, Φ is a transverse 2-form on $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$. It is degenerate with null space span$\{\xi_1, \xi_2\}$.

Also, using the calculus in local coordinates, we easily obtain

$$\Phi\left(\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b}\right) = 0, \qquad \Phi\left(\frac{\delta}{\delta x^a}, \frac{\partial}{\partial p_b}\right) = -\delta^b_a + \frac{p_a p^b}{K^2},$$
$$\Phi\left(\frac{\partial}{\partial p_a}, \frac{\partial}{\partial p_b}\right) = 0. \tag{30}$$

On the other hand, we have

$$d\omega^1\left(\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b}\right) = \frac{\delta}{\delta x^a}\left(\frac{p_b}{K}\right) - \frac{\delta}{\delta x^b}\left(\frac{p_a}{K}\right),$$
$$d\omega^1\left(\frac{\delta}{\delta x^a}, \frac{\partial}{\partial p_b}\right) = \frac{1}{K}\left(-\delta^b_a + \frac{p_a p^b}{K^2}\right), \tag{31}$$
$$d\omega^1\left(\frac{\partial}{\partial p_a}, \frac{\partial}{\partial p_b}\right) = 0,$$

where in the second relation we have used $p^a = K(\partial K/\partial p_a)$ which follows from the homogeneity conditions (18) of K. Comparing now Φ with $d\omega^1$ we obtain

$$\frac{1}{K}\Phi = d\omega^1 + \Psi, \tag{32}$$

where $\Psi = ((\delta/\delta x^b)(p_a/K) - (\delta/\delta x^a)(p_b/K))dx^a \wedge dx^b$. Thus, $(1/K)\Phi$ is transversally closed if and only if Ψ is transversally closed. Concluding, $(1/K)\Phi$ is in general an almost transversally presymplectic structure on $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$.

Similarly, for $\varepsilon = -1$, we can put

$$H(X,Y) = G(f(X), Y), \quad X, Y \in \Gamma(Q\widetilde{\mathcal{F}}). \tag{33}$$

We have the following.

Theorem 10. *The mapping H is a symmetric bilinear form on $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$ and the annihilator of H is* ker f.

Proof. The symmetry and bilinearity are obvious. Also, the null space of H is

$$\begin{aligned}&\left\{X \in \Gamma\left(Q\widetilde{\mathcal{F}}\right) \mid H(X, Y)=0,\ \forall Y \in \Gamma\left(Q\widetilde{\mathcal{F}}\right)\right\} \\ &\quad=\left\{X \in \Gamma\left(Q\widetilde{\mathcal{F}}\right) \mid G(f(X), Y)=0\right\}=\ker f\end{aligned} \tag{34}$$

which end the proof. □

Locally, we obtain

$$H=\left(g_{ab}-\frac{p_a p_b}{K^2}\right) dx^a \otimes dx^b-\left(g^{ab}-\frac{p^a p^b}{K^2}\right) \delta p_a \otimes \delta p_b \tag{35}$$

with $\det(g_{ab}-(p_a p_b/K^2))=0$, since $(g_{ab}-(p_a p_b/K^2))p^b = p_a - p_a = 0$, and similarly $\det(g^{ab}-(p^a p^b/K^2))=0$, since $(g^{ab}-(p^a p^b/K^2))p_b = p^a - p^a = 0$.

Remark 11. The map H is a transversally singular pseudo-Riemannian metric on $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$.

5. An Almost (Para)Contact Structure on Transverse Liouville Distribution of $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$

Denote by $\{\xi_2\}$ the line vector bundle over $Q^*\mathcal{F}$ spanned by ξ_2 and we define the transverse vertical Liouville distribution as the complementary orthogonal distribution $S(Q^*\mathcal{F})$ to $\{\xi_2\}$ in $V(Q^*\mathcal{F})$ with respect to G^v; namely, $V(Q^*\mathcal{F}) = S(Q^*\mathcal{F}) \oplus \{\xi_2\}$. Hence, $S(Q^*\mathcal{F})$ is defined by $\alpha := \omega^2|_{V(Q^*\mathcal{F})}$; that is,

$$\Gamma(S(Q^*\mathcal{F}))=\{X \in \Gamma(V(Q^*\mathcal{F}));\ \alpha(X)=0\}. \tag{36}$$

Thus, any transverse vertical vector field $X \in \Gamma(V(Q^*\mathcal{F}))$ can be expressed as

$$X=PX+\alpha(X)\xi_2, \tag{37}$$

where P is the projection morphism of $V(Q^*\mathcal{F})$ on $S(Q^*\mathcal{F})$. By direct calculations, one gets the following.

Proposition 12. *For any transverse vertical vector fields $X, Y \in \Gamma(V(Q^*\mathcal{F}))$, one has*

$$G^v(X, PY)=G^v(PX, PY)=G^v(X, Y)-\alpha(X)\alpha(Y). \tag{38}$$

Theorem 13. *The transverse vertical Liouville distribution $S(Q^*\mathcal{F})$ is integrable.*

Proof. The proof follows using an argument similar to Theorem 3.1 [14] (see also Theorem 2.1 [6] or Theorem 4 [15]). □

In the following, we will consider the transverse Liouville distribution of $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$ as the complementary orthogonal distribution $\overline{S}(Q^*\mathcal{F})$ to $\{\xi_2\}$ in $Q\widetilde{\mathcal{F}}$ with respect to G; that is, $\overline{S}(Q^*\mathcal{F}) = H(Q^*\mathcal{F}) \oplus S(Q^*\mathcal{F})$.

Let us restrict to $\overline{S}(Q^*\mathcal{F})$ all the geometrical structures introduced in Section 4 for all $Q\widetilde{\mathcal{F}}$. We indicate this by overlines. Hence, we have

(i) $\overline{\xi_1} = \xi_1$ since ξ_1 lies in $\overline{S}(Q^*\mathcal{F})$;

(ii) $\overline{\omega^2} = 0$ since $\omega^2(X) = G(X, \xi_2) = 0$ for every transverse vector field $X \in \overline{S}(Q^*\mathcal{F})$;

(iii) $\overline{G} = G|_{\overline{S}(Q^*\mathcal{F})}$;

(iv) $\overline{f}(X) = \overline{\phi}(X) + \varepsilon\overline{\omega^1}(X) \otimes \xi_2$ is an endomorphism of $\overline{S}(Q^*\mathcal{F})$ since

$$\begin{aligned}G\left(\overline{f}(X), \xi_2\right)&=G\left(\overline{\phi}(X), \xi_2\right)+\varepsilon\overline{\omega^1}(X) G(\xi_2, \xi_2) \\ &=\omega^2\left(\overline{\phi}(X)\right)+\varepsilon\overline{\omega^1}(X)=0.\end{aligned} \tag{39}$$

We denote now $\overline{\xi} = \overline{\xi_1}$ and $\overline{\eta} = \overline{\omega^1}$.

By Theorem 8, we obtain the following.

Theorem 14. *The triple $(\overline{f}, \overline{\xi}, \overline{\eta})$ provides an almost (para)contact structure on $\overline{S}(Q^*\mathcal{F})$; that is,*

(i) $\overline{f}^3 + \varepsilon\overline{f} = 0$, *rank* $\overline{f} = 2n-2 = (2n-1)-1$;

(ii) $\overline{\eta}(\overline{\xi}) = 1$, $\overline{f}(\overline{\xi}) = 0$, $\overline{\eta} \circ \overline{f} = 0$;

(iii) $\overline{f}^2(X) = -\varepsilon(X - \overline{\eta}(X)\overline{\xi})$, *for* $X \in \overline{S}(Q^*\mathcal{F})$.

Also, by Theorem 9 we obtain the following.

Theorem 15. *The transversal Riemannian metric $\overline{G}$ verifies*

$$\overline{G}\left(\overline{f}(X), \overline{f}(Y)\right)=\overline{G}(X, Y)-\overline{\eta}(X)\overline{\eta}(Y), \tag{40}$$

for every transverse vector fields $X, Y \in \overline{S}(Q^*\mathcal{F})$.

Concluding, the ensemble $(\overline{f}, \overline{\xi}, \overline{\eta}, \overline{G})$ is an almost (para)contact Riemannian structure on $\overline{S}(Q^*\mathcal{F})$.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

Both authors contributed equally to the paper. All the authors read and approved the final paper.

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