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Research Article

Z-Distance Based IF-THEN Rules

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Decision making, reasoning, and analysis in real-world problems are complicated by imperfect information. Real-world imperfect information is mainly characterized by two features. In view of this, Professor Zadeh suggested the concept of a Z-number as an ordered pair $Z = (A, B)$ of fuzzy numbers A and B, the first of which is a linguistic value of a variable of interest, and the second one is a linguistic value of probability measure of the first one, playing a role of its reliability. The concept of distance is one of the important concepts for handling imperfect information in decision making and reasoning. In this paper, we, for the first time, apply the concept of distance of Z-numbers to the approximate reasoning with Z-number based IF-THEN rules. We provide an example on solving problem related to psychological issues naturally characterized by imperfect information, which shows applicability and validity of the suggested approach.

1. Introduction

Decision making, reasoning, and analysis in real-world problems are complicated by imperfect information. Real-world imperfect information is mainly characterized by two features. On the one hand, real-world information is often described on a basis of perception, experience, and knowledge of a human being. In turn, these operate with linguistic description carrying imprecision and vagueness, for which fuzzy sets based formalization can be used. On the other side, perception, experience, and knowledge of a human being are not sources of the truth. Therefore, the reliability is a degree of a partial confidence of a human being, which is naturally partial. This partial reliability is also naturally imprecise and can be formalized as a fuzzy value of probability measure. In order to ground the formal basis for dealing with real-world information, Zadeh suggested the concept of a Z-number [1] as an ordered pair $Z = (A, B)$ of continuous fuzzy numbers used to describe a value of a random variable X, where A is a fuzzy constraint on values of X and B is a fuzzy reliability of A and is considered as a value of probability measure of A. Nowadays a series of works devoted to Z-numbers and their application in decision making, control, and other fields [2–13] exists. A general and computationally effective approach to computation with discrete Z-numbers is suggested in [14–16]. The authors provide motivation of the use of discrete Z-numbers mainly based on the fact that NL-based information is of a discrete framework. The suggested arithmetic of discrete Z-numbers includes basic arithmetic operations and important algebraic operations.

The concept of distance is one of the important concepts for decision making and reasoning [17, 18]. In this paper, we for the first time apply the concept of distance of Z-numbers to the approximate reasoning with Z-number based IF-THEN rules. An approximate reasoning refers to a process of inferring imprecise conclusions from imprecise premises [17–38]. As one can see, this process often takes place in various fields of human activity including economics, decision analysis, system analysis, control, and everyday activity. The reason for this is that information relevant to real-world problems is, as a rule, imperfect. According to Zadeh, imperfect information is information which in one or more respects is imprecise, uncertain, incomplete, unreliable, vague, or partially true [39]. We can say that in a wide sense approximate reasoning is reasoning with imperfect information.

The paper is structured as follows. In Section 2, we present some prerequisite material including definitions of a discrete fuzzy number, a discrete Z-number, and probability measure of a discrete fuzzy number. In Section 3, we propose several distance measures for Z-numbers. In Section 4, we describe the statement of the problem and the suggested approach to reasoning with Z-rules on the basis of distance of Z-numbers. In Section 5, we illustrate an application of the suggested approach to a real-world problem which involves modeling of psychological aspects. Section 6 concludes.

2. Preliminaries

2.1. Main Definitions

Definition 1 (a discrete fuzzy number [40–43]). A fuzzy subset A of the real line $\mathscr{R}$ with membership function $\mu_A : \mathscr{R} \to [0,1]$ is a discrete fuzzy number if its support is finite; that is, there exist $x_1, \ldots, x_n \in \mathscr{R}$ with $x_1 < x_2 < \cdots < x_n$, such that $\operatorname{supp}(A) = \{x_1, \ldots, x_n\}$ and there exist natural numbers s, t with $1 \le s \le t \le n$ satisfying the following conditions:

(1) $\mu_A(x_i) = 1$ for any natural number i with $s \le i \le t$;

(2) $\mu_A(x_i) \le \mu_A(x_j)$ for natural numbers i, j with $1 \le i \le j \le s$;

(3) $\mu_A(x_i) \ge \mu_A(x_j)$ for natural numbers i, j with $t \le i \le j \le n$.

Definition 2 (a discrete random variable and a discrete probability distribution [44]). A random variable, X, is a variable whose possible values x are outcomes of a random phenomenon. A discrete random variable is a random variable which takes only a countable set of its values x.

Consider a discrete random variable X with outcomes space $\{x_1, \ldots, x_n\}$. A probability of an outcome $X = x_i$, denoted $P(X = x_i)$, is defined in terms of a probability distribution. A function p is called a discrete probability distribution or a probability mass function if

$$P(X = x_i) = p(x_i), \tag{1}$$

where $p(x_i) \in [0,1]$ and $\sum_{i=1}^{n} p(x_i) = 1$.

Definition 3 (arithmetic operations over discrete random variables [44, 45]). Let X_1 and X_2 be two independent discrete random variables with the corresponding outcome spaces $X_1 = \{x_{11}, \ldots, x_{1i}, \ldots, x_{1n_1}\}$ and $X_2 = \{x_{21}, \ldots x_{2i}, \ldots, x_{2n_2}\}$ and the corresponding discrete probability distributions p_1 and p_2. The probability distribution of $X_{12} = X_1 * X_2$, $* \in \{+, -, \cdot, /\}$, is the convolution $p_{12} = p_1 \circ p_2$ of p_1 and p_2 which is defined for any $x \in \{x_1 * x_2 \mid x_1 \in X_1,\ x_2 \in X_2\}$, $x_1 \in X_1, x_2 \in X_2$, as follows:

$$p_{12}(x) = \sum_{x = x_1 * x_2} p_1(x_1)\, p_2(x_2). \tag{2}$$

Definition 4 (probability measure of a discrete fuzzy number [46]). Let X be discrete random variable with probability distribution p. Let A be a discrete fuzzy number describing a possibilistic restriction on values of X. A probability measure of A denoting $P(A)$ is defined as

$$\begin{aligned} P(A) &= \sum_{i=1}^{n} \mu_A(x_i)\, p(x_i) \\ &= \mu_A(x_1)\, p(x_1) + \mu_A(x_2)\, p(x_2) + \cdots \\ &\quad + \mu_A(x_n)\, p(x_n). \end{aligned} \tag{3}$$

Definition 5 (a scalar multiplication of a discrete fuzzy number [16]). A scalar multiplication of a discrete fuzzy number A by a real number $\lambda \in \mathscr{R}$ is the discrete fuzzy number $A_1 = \lambda A$, whose α-cut is defined as

$$\begin{aligned} &A_1^{\alpha} \\ &\quad = \{x \in \lambda \cdot \operatorname{supp}(A) \mid \min(\lambda A^{\alpha}) \le x \le \max(\lambda A^{\alpha})\}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \lambda \cdot \operatorname{supp}(A) &= \{\lambda x \mid x \in \operatorname{supp}(A)\}, \\ \min(\lambda A^{\alpha}) &= \min\{\lambda x \mid x \in A^{\alpha}\}, \\ \max(\lambda A^{\alpha}) &= \max\{\lambda x \mid x \in A^{\alpha}\}, \end{aligned} \tag{5}$$

and the membership function is defined as

$$\mu_{\lambda A}(x) = \sup\{\alpha \in [0,1] \mid x \in (\lambda A^{\alpha})\}. \tag{6}$$

Definition 6 (addition of discrete fuzzy numbers [40–43]). For discrete fuzzy numbers A_1, A_2, their addition $A_{12} = A_1 + A_2$ is the discrete fuzzy number whose α-cut is defined as

$$\begin{aligned} A_{12}^{\alpha} = \{x \in \{\operatorname{supp}(A_1) + \operatorname{supp}(A_2)\} \mid \min\{A_1^{\alpha} + A_2^{\alpha}\} \\ \le x \le \max\{A_1^{\alpha} + A_2^{\alpha}\}\}, \end{aligned} \tag{7}$$

where $\operatorname{supp}(A_1) + \operatorname{supp}(A_2) = \{x_1 + x_2 \mid x_j \in \operatorname{supp}(A_j),\ j = 1,2\}$, $\min\{A_1^{\alpha} + A_2^{\alpha}\} = \min\{x_1 + x_2 \mid x_j \in A_j^{\alpha},\ j = 1,2\}$, $\max\{A_1^{\alpha} + A_2^{\alpha}\} = \max\{x_1 + x_2 \mid x_j \in A_j^{\alpha},\ j = 1,2\}$, and the membership function is defined as

$$\mu_{A_1 + A_2}(x) = \sup\{\alpha \in [0,1] \mid x \in \{A_1^{\alpha} + A_2^{\alpha}\}\}. \tag{8}$$

Definition 7 (a discrete Z-number [15, 16]). A discrete Z-number is an ordered pair $Z = (A, B)$ of discrete fuzzy numbers A and B. A plays a role of a fuzzy constraint on values that a random variable X may take. B is a discrete fuzzy number with a membership function $\mu_B : \{b_1, \ldots, b_n\} \to [0,1]$, $\{b_1, \ldots, b_n\} \subset [0,1]$, playing a role of a fuzzy constraint on the probability measure of A, $P(A) = \sum_{i=1}^{n} \mu_A(x_i) p(x_i)$, $P(A) \in \operatorname{supp}(B)$.

3. Distance between Two Z-Numbers

Denote by $\mathscr{F}$ the space of discrete fuzzy sets of $\mathscr{R}$. Denote by $\mathscr{F}_{[a,b]}$ the space of discrete fuzzy sets of $[a,b] \subset \mathscr{R}$.

Definition 8 (the supremum metric on $\mathscr{D}$ [47]). The supremum metric d on $\mathscr{F}$ is defined as

$$d(A_1, A_2) = \sup\{d_H(A_1^\alpha, A_2^\alpha) \mid 0 < \alpha \le 1\}, \quad A_1, A_2 \in \mathscr{F}, \tag{9}$$

where d_H is the Hausdorff distance.

$(\mathscr{F}, d)$ is a complete metric space [47, 48].

Definition 9 (fuzzy Hausdorff distance [16]). The fuzzy Hausdorff distance d_{fH} between $A_1, A_2 \in \mathscr{F}$ is defined as

$$d_{fH}(A_1, A_2) = \bigcup_{\alpha\in[0,1]} \alpha d_{fH}^\alpha (A_1, A_2), \tag{10}$$

where

$$d_{fH}^\alpha (A_1, A_2) = \left\{ \sup_{\alpha\le\overline{\alpha}\le 1} d_H\left(A_1^{\overline{\alpha}}, A_2^{\overline{\alpha}}\right) \right\}, \tag{11}$$

where $\overline{\alpha}$ is the value which is within α-cut and 1-cut. $(\mathscr{F}, d_{fH})$ is a complete metric space.

Denote by $\mathscr{Z}$ the space of discrete Z-numbers:

$$\mathscr{Z} = \{Z = (A, B) \mid A \in \mathscr{F},\ B \in \mathscr{F}_{[0,1]}\}. \tag{12}$$

Definition 10 (supremum metrics on $\mathscr{Z}$ [16]). The supremum metrics on $\mathscr{Z}$ are defined as

$$D(Z_1, Z_2) = d(A_1, A_2) + d(B_1, B_2); \tag{13}$$

$(\mathscr{Z}, D)$ is a complete metric space. This follows from the fact that $(\mathscr{F}, d)$ is a complete metric space.

$D(Z_1, Z_2)$ has the following properties:

$$\begin{gathered} D(Z_1 + Z, Z_2 + Z) = D(Z_1, Z_2), \\ D(Z_2, Z_1) = D(Z_1, Z_2), \\ D(\lambda Z_1, \lambda Z_2) = |\lambda| D(Z_1, Z_2), \quad \lambda \in \mathscr{R}, \\ D(Z_1, Z_2) \le D(Z_1, Z) + D(Z, Z_2). \end{gathered} \tag{14}$$

Definition 11 (fuzzy Hausdorff distance between Z-numbers [16]). The fuzzy Hausdorff distance d_{fHZ} between Z-numbers $Z_1 = (A_1, B_1), Z_2 = (A_2, B_2) \in \mathscr{Z}$ is defined as

$$d_{fHZ}(Z_1, Z_2) = d_{fH}(A_1, A_2) + d_{fH}(B_1, B_2). \tag{15}$$

Definition 12 (Z-valued Euclidean distance between discrete Z-numbers [16]). Given two discrete Z-numbers $Z_1 = (A_1, B_1), Z_2 = (A_2, B_2) \in \mathscr{Z}$, Z-valued Euclidean distance $d_E(Z_1, Z_2)$ between Z_1 and Z_2 is defined as

$$d_E(Z_1, Z_2) = \sqrt{(Z_1 - Z_2)^2}. \tag{16}$$

4. Z-Valued IF-THEN Rules Based Reasoning

A problem of interpolation of Z-rules termed as Z-interpolation was addressed by Zadeh as a challenging problem [33]. This problem is the generalization of interpolation of fuzzy rules [49]. The problem of Z-interpolation is given below.

Given the following Z-rules,

if X_1 is $Z_{X_1,1} = (A_{X_1,1}, B_{X_1,1})$ and so on and X_m is $Z_{X_m,1} = (A_{X_m,1}, B_{X_m,1})$, then Y is $Z_Y = (A_{Y,1}, B_{Y,1})$,

if X_1 is $Z_{X_1,2} = (A_{X_1,2}, B_{X_1,2})$ and so on and X_m is $Z_{X_m,2} = (A_{X_m,2}, B_{X_m,2})$, then Y is $Z_Y = (A_{Y,2}, B_{Y,2})$,

if X_1 is $Z_{X_1,n} = (A_{X_1,n}, B_{X_1,n})$ and so on and X_m is $Z_{X_m,n} = (A_{X_m,n}, B_{X_m,n})$ then Y is $Z_Y = (A_{Y,n}, B_{Y,n})$,

and a current observation

X_1 is $Z'_{X_1} = (A'_{X_1}, B'_{X_1})$ and so on and X_m is $Z'_{X_m} = (A'_{X_m}, B'_{X_m})$,

find the Z-value of Y. Here m is the number of Z-valued input variables and n is the number of rules.

The idea underlying the suggested interpolation approach is that the ratio of distances between the resulting output and the consequent parts is equal to one between the current input and the antecedent parts [49]. This implies for Z-rules that the resulting output Z'_Y is computed as

$$Z'_Y = \sum_{j=1}^{n} w_j Z_{Y,j} = \sum_{j=1}^{n} w_j \left(A_{Y,j}, B_{Y,j}\right), \tag{17}$$

where $Z_{Y,j}$ is the Z-number valued consequent of the jth rule, $w_j = (1/\rho_j)/(\sum_{j=1}^{n} 1/\rho_j)$, $j = 1, \ldots, n$ are coefficients of linear interpolation, and n is the number of Z-rules. $\rho_j = \sum_{i=1}^{m} D(Z'_{X_i}, Z_{X_i,j})$, where D is the distance between current ith Z-number valued input and the ith Z-number valued antecedent of the jth rule. Thus, ρ_j computes the distance between a current input vector and the vector of the antecedents of jth rule.

In this paper, we will consider discrete Z-numbers. The operations of addition and scalar multiplication of discrete Z-numbers are described below.

Addition of Discrete Z-Numbers. Let $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ be discrete Z-numbers describing imperfect information about values of variables X_1 and X_2. Consider the problem of computation of addition $Z_{12} = Z_1 + Z_2$. The first stage is the computation addition of discrete fuzzy numbers $A_{12} = A_1 + A_2$ on the basis of Definition 6.

The second stage involves stage-by-stage construction of B_{12} which is related to propagation of probabilistic restrictions. We realize that, in Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$, the "true" probability distributions p_1 and p_2 are not exactly known. In contrast, the information available is represented by the fuzzy restrictions:

$$\begin{gathered} \sum_{k=1}^{n_1} \mu_{A_1}(x_{1k})\, p_1(x_{1k}) \text{ is } B_1, \\ \sum_{k=1}^{n_2} \mu_{A_2}(x_{2k})\, p_2(x_{2k}) \text{ is } B_2, \end{gathered} \tag{18}$$

which are represented in terms of the membership functions as

$$\mu_{B_1}\left(\sum_{k=1}^{n_1}\mu_{A_1}(x_{1k})\,p_1(x_{1k})\right),\qquad \mu_{B_2}\left(\sum_{k=1}^{n_2}\mu_{A_2}(x_{2k})\,p_2(x_{2k})\right). \tag{19}$$

Thus, one has the fuzzy sets of probability distributions of p_1 and p_2 with the membership functions defined as

$$\mu_{p_1}(p_1)=\mu_{B_1}\left(\sum_{k=1}^{n_1}\mu_{A_1}(x_{1k})\,p_1(x_{1k})\right),\qquad \mu_{p_2}(p_2)=\mu_{B_2}\left(\sum_{k=1}^{n_2}\mu_{A_2}(x_{2k})\,p_2(x_{2k})\right). \tag{20}$$

Therefore, we should construct these fuzzy sets. B_j, $j=1,2$, is a discrete fuzzy number which plays the role of a soft constraint on a value of a probability measure of A_j. Therefore, basic values $b_{jl}\in\operatorname{supp}(B_j)$, $j=1,2,l=1,\ldots,m$, of a discrete fuzzy number B_j, $j=1,2$, are values of a probability measure of A_j, $b_{jl}=P(A_j)$. Thus, given b_{jl}, we have to find such probability distribution p_{jl} which satisfies

$$b_{jl}=\mu_{A_j}(x_{j1})\,p_{jl}(x_{j1})+\mu_{A_j}(x_{j2})\,p_{jl}(x_{j2})+\cdots+\mu_{A_j}(x_{jn_j})\,p_{jl}(x_{jn_j}). \tag{21}$$

At the same time, we know that p_{jl} has to satisfy

$$\sum_{k=1}^{n_j}p_{jl}(x_{jk})=1,\quad p_{jl}(x_{jk})\geq 0. \tag{22}$$

Thus, the following goal programming problem should be solved to find p_j:

$$\mu_{A_j}(x_{j1})\,p_{jl}(x_{j1})+\mu_{A_j}(x_{j2})\,p_{jl}(x_{j2})+\cdots+\mu_{A_j}(x_{jn_j})\,p_{jl}(x_{jn_j})\longrightarrow b_{jl}, \tag{23}$$

subject to

$$p_{jl}(x_{j1})+p_{jl}(x_{j2})+\cdots+p_{jl}(x_{jn_j})=1,\qquad p_{jl}(x_{j1}),p_{jl}(x_{j2}),\ldots,p_{jl}(x_{jn_j})\geq 0. \tag{24}$$

For each $l=1,\ldots,m$ and each $k=1,\ldots,n_j$ denote $c_k=\mu_{A_j}(x_{jk})$ and $v_k^l=p_{jl}(x_{jk})$, $k=1,\ldots,n_j$. As c_k and b_{jl} are known and v_k^l are unknown, we see that problem (23)-(24) is nothing but the following goal linear programming problem:

$$c_1v_1^l+c_2v_2^l+\cdots+c_nv_n^l\longrightarrow b_{jl}, \tag{23$'$}$$

subject to

$$v_1^l+v_2^l+\cdots+v_n^l=1,\qquad v_1^l,v_2^l,\ldots,v_n^l\geq 0. \tag{24$'$}$$

Having obtained the solution v_k^l, $k=1,\ldots,n_j$, of problems (23′)-(24′) for each $l=1,\ldots,m$, recall that $v_k^l=p_{jl}(x_{jk})$, $k=1,\ldots,n_j$. As a result, $p_{jl}(x_{jk})$, $k=1,\ldots,n_j$, is found, and, therefore, distribution p_{jl} is obtained. Thus, to construct $\mu_{p_{jl}}$, we need to solve m simple problems (23′)-(24′). Let us mention that in general, problems (23′)-(24′) do not have a unique solution. In order to guarantee existence of a unique solution, the compatibility conditions can be included:

$$\sum_{k=1}^{n_j}x_{jk}p_{jl}(x_{jk})=\frac{\sum_{k=1}^{n_j}x_{jk}\mu_{A_j}(x_{jk})}{\sum_{k=1}^{n_j}\mu_{A_j}(x_{jk})}. \tag{25}$$

This condition implies that the centroid of A_j is to coincide with that of p_{jl}.

Probability distributions $p_{jl}(x_{jk})$, $k=1,\ldots,n_j$, naturally induce probabilistic uncertainty over the result $X=X_1+X_2$. This implies, given any possible pair p_{1l},p_{2l} of the extracted distributions, the convolution $p_{12s}=p_{1l}\circ p_{2l}$, $s=1,\ldots,m^2$, is to be computed as follows:

$$p_{12}(x)=\sum_{x_1+x_2=x}p_{1l}(x_1)\,p_{2l}(x_2),\qquad \forall x\in X_{12};\ x_1\in X_1,\ x_2\in X_2. \tag{26}$$

Given p_{12s}, the value of probability measure of A_{12} can be computed:

$$P(A_{12})=\sum_{k=1}^{n}\mu_{A_{12}}(x_{12k})\,p_{12}(x_{12k}). \tag{27}$$

However, the "true" p_{12s} is not exactly known as the "true" p_{1l} and p_{2l} are described by fuzzy restrictions. In other words, the fuzzy sets of probability distributions p_{1l} and p_{2l} induce the fuzzy set of convolutions p_{12s}, $s=1,\ldots,m^2$, with the membership function defined as

$$\mu_{p_{12}}(p_{12})=\max_{p_1,p_2}\left[\mu_{p_1}(p_1)\wedge\mu_{p_2}(p_2)\right], \tag{28}$$

subject to

$$p_{12}=p_1\circ p_2,\qquad \mu_{p_j}(p_j)=\mu_{B_j}\left(\sum_{k=1}^{n_j}\mu_{A_j}(x_{jk})\,p_{jl}(x_{jk})\right), \tag{29}$$

where $\wedge$ is min operation.

As a result, fuzziness of information on p_{12s} described by $\mu_{p_{12}}$ induces fuzziness of the value of probability measure $P(A_{12})$ as a discrete fuzzy number B_{12}. The membership function $\mu_{B_{12}}$ is defined as

$$\mu_{B_{12}}(b_{12s})=\sup\left(\mu_{p_{12s}}(p_{12s})\right), \tag{30}$$

subject to

$$b_{12s} = \sum_k p_{12s}(x_k)\,\mu_{A_{12}}(x_k). \quad (31)$$

As a result, $Z_{12} = Z_1 + Z_2$ is obtained as $Z_{12} = (A_{12}, B_{12})$.

Scalar Multiplication of Discrete Z-Numbers. Let us consider a scalar multiplication of a discrete Z-number $Z_X = (A_X, B_X)$: $Z_Y = \lambda \cdot Z_X$, $\lambda \in \mathscr{R}$. The resulting $Z_Y = (A_Y, B_Y)$ is found as follows. $A_Y = \lambda A_X$ is determined based on Definition 5.

In order to construct B_Y, at first probability distributions $p_{X,l}$, $l = 1, \ldots, m$, should be extracted by solving a linear programming problem analogous to (23′)-(24′). Next, we realize that $p_{X,l}$, $l = 1, \ldots, m$, induce probability distributions $p_{Y,l}$, $l = 1, \ldots, m$, related to Z_Y as follows:

$$p_Y = p_Y(y_1) \setminus y_1 + p_Y(y_2) \setminus y_2 + \cdots + p_Y(y_n) \setminus y_n, \quad (32)$$

such that

$$\begin{gathered} y_k = \lambda x_k, \\ p_Y(y_k) = p_X(x_k). \end{gathered} \quad (33)$$

The fuzzy set of probability distributions p_X with membership function $\mu_{p_X}(p_{X,l}) = \mu_{\widetilde{B}_X}(\sum_{k=1}^{n} \mu_{\widetilde{A}_X}(x_k) p_{X,l}(x_k))$ induces the fuzzy set of probability distributions $p_{Y,l}$ with the membership function defined as

$$\mu_{p_Y}(p_{Y,l}) = \mu_{p_X}(p_{X,l}), \quad (34)$$

taking into account (32)-(33).

Next, we compute probability measure of A_Y, given p_Y. Given a fuzzy restriction on p_Y described by μ_{p_Y}, we construct a fuzzy number B_Y with the membership function μ_{B_Y}:

$$\mu_{B_Y}(b_{Y,l}) = \sup\left(\mu_{p_Y}(p_{Y,l})\right), \quad (35)$$

subject to

$$b_{Y,l} = \sum_k p_{Y,l}(x_k)\,\mu_{A_Y}(x_k). \quad (36)$$

As a result, $Z_Y = \lambda \cdot Z_X$ is obtained as $Z_Y = (A_Y, B_Y)$.

Let us now consider the special case of the considered problem of Z-rules interpolation, suggested in [50, 51].

Given the Z-rules

$$\begin{gathered} \text{If } X \text{ is } A_{X,1} \text{ then } Y \text{ is } (A_{Y,1}, B) \\ \text{If } X \text{ is } A_{X,2} \text{ then } Y \text{ is } (A_{Y,2}, B) \\ \vdots \\ \text{If } X \text{ is } A_{X,n} \text{ then } Y \text{ is } (A_{Y,n}, B) \end{gathered} \quad (37)$$

and a current observation

$$X \text{ is } (A_X, B_X), \quad (38)$$

find the Z-value of Y.

For this case, as the reliabilities of the Z-number based consequents of the considered rules are equal, $B_{Y,k} = B$, according to formula (17) the Z-number valued output of the Z-rules, $Z'_Y = (A'_Y, B'_Y)$, is computed as

$$\begin{aligned} Z'_Y &= \sum_{j=1}^{n} w_j Z_{Y,j} = \sum_{j=1}^{n} w_j \left(A_{Y,j}, B_{Y,j}\right) \\ &= \sum_{j=1}^{n} w_j \left(A_{Y,j}, B\right), \end{aligned} \quad (39)$$

where $w_j = (1/\rho_j)/(\sum_{k=1}^{n} 1/\rho_k)$ and $\rho_j = \sum_{i=1}^{m} D(Z'_{X_i}, Z_{X_i,j}) = \sum_{i=1}^{m} D((A'_{X_i}, 1), (A_{X_i,j}, 1)) = \sum_{i=1}^{m} d(A'_{X_i}, A_{X_i,j})$ as both inputs and the antecedents of the considered Z-rules are of a special Z-number; that is, they are represented by discrete fuzzy numbers with the reliability equal to 1.

5. An Application

Let us consider modeling of a fragment of a relationship between the student motivation, attention, anxiety, and educational achievement [52]. The information on the considered characteristics is naturally imprecise and partially reliable. Indeed, one deals mainly with intangible, nonmeasurable mental indicators. For this reason, the use of Z-rules, as rules with Z-number valued inputs and outputs based on linguistic terms from a predefined codebook, is adequate way for modeling of this relationship. This rules will help to evaluate a student with given Z-number based evaluations of the characteristics. Consider the following Z-rules:

> *The 1st rule*: If motivation is (M, U), attention is (H, U), and anxiety is (L, U), then achievement is (E, U).
>
> *The 2nd rule*: If motivation is (M, U), attention is (M, U), and anxiety is (M, U), then achievement is (G, U).

Here, the pairs $(\cdot,\cdot)$ are Z-numbers where uppercase letters denote the following linguistic terms: H, High; L, Low; M, Medium; G, Good; E, Excellence; U, Usually. The codebooks containing linguistic terms of values of antecedents and consequents are given in Figures 1, 2, 3, and 4. The codebook for the degrees of reliability of values of antecedents and consequents is shown in Figure 5.

The considered Z-numbers are given below.

The 1st rule inputs:

$$\begin{aligned} Z_{A_M} &= \frac{0}{2.6} + \frac{0.5}{3.3} + \frac{1}{4} + \frac{0.5}{4.7} + \frac{0}{5.4}, \\ Z_{B_U} &= \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}; \\ Z_{A_H} &= \frac{0}{57.5} + \frac{0.5}{68.75} + \frac{1}{80} + \frac{1}{90}, \end{aligned}$$

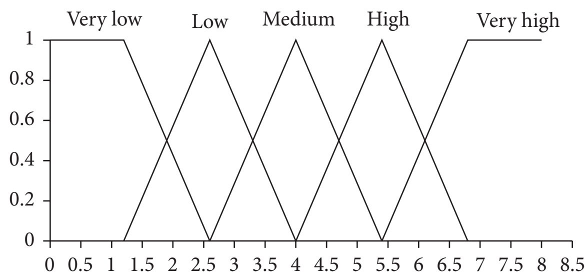

Figure 1: Linguistic terms for a value of motivation.

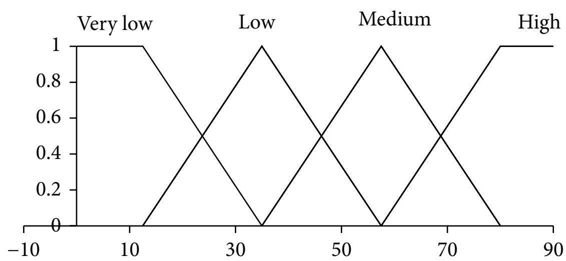

Figure 2: Linguistic terms for a value of attention.

$$Z_{B_U} = \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9};$$
$$Z_{A_L} = \frac{0}{1.19} + \frac{0.5}{1.6} + \frac{1}{2} + \frac{0.5}{2.4} + \frac{0}{2.8},$$
$$Z_{B_U} = \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}. \tag{40}$$

The 1st rule output:

$$Z_{A_{VH}} = \frac{0}{80} + \frac{0.5}{85} + \frac{1}{90} + \frac{0.5}{95} + \frac{0}{100},$$
$$Z_{B_U} = \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}. \tag{41}$$

The 2nd rule inputs:

$$Z_{A_M} = \frac{0}{2.6} + \frac{0.5}{3.3} + \frac{1}{4} + \frac{0.5}{4.7} + \frac{0}{5.4},$$
$$Z_{B_U} = \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9};$$
$$Z_{A_M} = \frac{0}{35} + \frac{0.5}{46.25} + \frac{1}{57.5} + \frac{0.5}{68.75} + \frac{0}{80},$$
$$Z_{B_U} = \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}; \tag{42}$$
$$Z_{A_M} = \frac{0}{2} + \frac{0.5}{2.4} + \frac{1}{2.8} + \frac{0.5}{3.2} + \frac{0}{3.6},$$
$$Z_{B_U} = \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}.$$

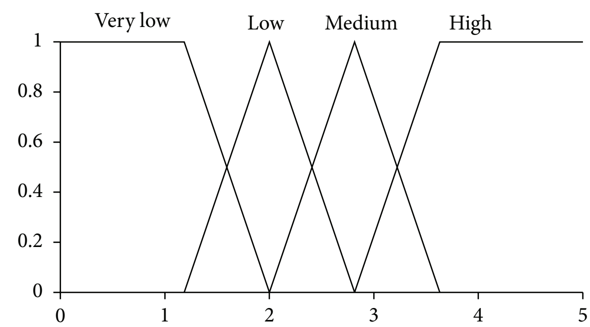

Figure 3: Linguistic terms for a value of anxiety.

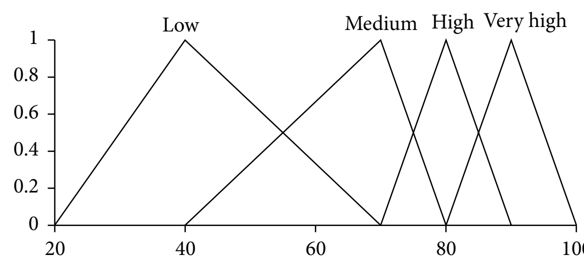

Figure 4: Linguistic terms for a value of achievement.

The 2nd rule output:

$$Z_{A_H} = \frac{0}{70} + \frac{0.5}{75} + \frac{1}{80} + \frac{0.5}{85} + \frac{0}{90},$$
$$Z_{B_U} = \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}. \tag{43}$$

Consider a problem of reasoning within the given Z-rules by using the suggested Z-interpolation approach. Let the current input information for motivation, attention, and anxiety be described by the following Z-numbers $Z_1 = (Z_{A_1}, Z_{B_1})$, $Z_2 = (Z_{A_2}, Z_{B_2})$, and $Z_3 = (Z_{A_3}, Z_{B_3})$, respectively:

$$Z_{A_1} = \frac{0}{2.5} + \frac{0.5}{3} + \frac{1}{3.5} + \frac{0.5}{4} + \frac{0}{4.5},$$
$$Z_{B_1} = \frac{0}{0.6} + \frac{0.5}{0.65} + \frac{1}{0.7} + \frac{0.5}{0.75} + \frac{0}{0.8};$$
$$Z_{A_2} = \frac{0}{25} + \frac{0.5}{35} + \frac{1}{45} + \frac{0.5}{55} + \frac{0}{65},$$
$$Z_{B_2} = \frac{0}{0.6} + \frac{0.5}{0.65} + \frac{1}{0.7} + \frac{0.5}{0.75} + \frac{0}{0.8}; \tag{44}$$
$$Z_{A_3} = \frac{0}{1.3} + \frac{0.5}{2.3} + \frac{1}{3.3} + \frac{0.5}{3.65} + \frac{0}{4},$$
$$Z_{B_3} = \frac{0}{0.6} + \frac{0.5}{0.65} + \frac{1}{0.7} + \frac{0.5}{0.75} + \frac{0}{0.8}.$$

Z-interpolation approach based reasoning consists of two main stages.

(1) For each rule compute *dist* as distance ρ_j between the current input Z-information $Z_1 = (Z_{A_1}, Z_{B_1})$, $Z_2 = (Z_{A_2}, Z_{B_2})$, and $Z_3 = (Z_{A_3}, Z_{B_3})$ and Z-antecedents of Z-rules base $Z_{j1} = (A_{j1}, B_{j1})$, $Z_{j2} = (A_{j2}, B_{j2})$, and $Z_{j3} =$

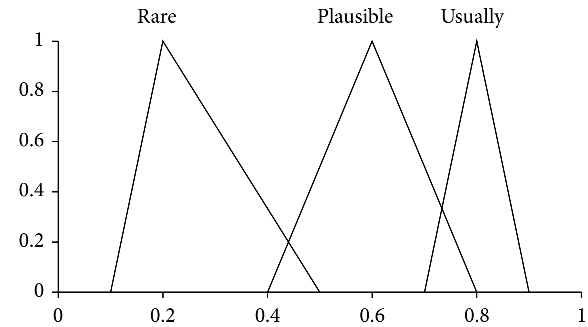

Figure 5: Linguistic terms for reliability of antecedents and consequents.

(A_{j3}, B_{j3}), $j = 1, 2$. For simplicity, we will use the supremum metric $D(Z_i, Z_{ji})$ (13):

$$\rho_j = \sum_{i=1}^{3} D\left(Z_i, Z_{ji}\right). \quad (45)$$

Consider computation of ρ_j for the 1st and 2nd rules. Thus, we need to determine $\rho_j = \sum_{j=1}^{3} D(Z_j, Z_{1j})$, where values $D(Z_1, Z_{11})$, $D(Z_2, Z_{12})$, and $D(Z_3, Z_{13})$ are computed on the basis of (13). We have obtained the results:

$$\begin{aligned} D\left(Z_1, Z_{11}\right) &= d_H\left(A_1, A_{11}\right) + d_H\left(B_1, B_{11}\right) \\ &= 0.9 + 0.1 = 1, \\ D\left(Z_2, Z_{12}\right) &= 40.1, \\ D\left(Z_3, Z_{13}\right) &= 1.4. \end{aligned} \quad (46)$$

Thus, the distance for the 1st rule is

$$\rho_1 = 42.5. \quad (47)$$

Analogously, we computed the distance for the 2nd rule as

$$\begin{aligned} D\left(Z_1, Z_{2,1}\right) &= 1, \\ D\left(Z_2, Z_{2,2}\right) &= 15.1, \\ D\left(Z_3, Z_{2,3}\right) &= 0.8, \\ \rho_2 &= 16.9. \end{aligned} \quad (48)$$

(2) Computation of the aggregated output Z_Y for Z-rules base by using linear Z-interpolation:

$$\begin{aligned} Z_Y &= w_1 Z_{Y,1} + w_2 Z_{Y,2}, \\ w_1 &= \frac{1/\rho_1}{1/\rho_1 + 1/\rho_2}, \quad w_2 = \frac{1/\rho_2}{1/\rho_1 + 1/\rho_2}. \end{aligned} \quad (49)$$

The obtained interpolation coefficients are $w_1 = 0.28$ and $w_2 = 0.72$. The aggregated output Z_Y is defined as

$$Z_Y = 0.28 Z_{Y_1} + 0.72 Z_{Y_2} = \left(A_Y, B_Y\right). \quad (50)$$

We have obtained the following result:

$$\begin{aligned} Z_{A_Y} &= \frac{0}{72.8} + \frac{0.5}{78.2} + \frac{1}{82.6} + \frac{0.5}{84} + \frac{0}{89}, \\ Z_{B_Y} &= \frac{0}{0.68} + \frac{0.5}{0.73} + \frac{1}{0.78} + \frac{0.5}{0.81} + \frac{0}{0.84}. \end{aligned} \quad (51)$$

In accordance with the codebooks shown in Figures 4 and 5, we have achievement is "*High*" with the reliability being "*Usually*." This linguistic approximation is made by using similarity measure between the obtained output and fuzzy sets in the codebooks.

6. Conclusion

A concept of a Z-number suggested by Zadeh is a key to computation with imprecise and partial reliable information. In this paper, we propose applying distance of Z-numbers to approximate reasoning within IF-THEN rules with Z-numbers-based antecedents and consequents.

A real-world application of the suggested research has been provided to illustrate its validity and potential applicability.

Competing Interests

The authors declare that they have no competing interests.

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