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Research Article

Analytical Approximant to a Quadratically Damped Duffing Oscillator

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The Duffing oscillator of a system with strong quadratic damping is considered. We give an elementary approximate analytical solution to this oscillator in terms of exponential and trigonometric functions. We compare the analytical approximant with the Runge–Kutta numerical solution. We also solve the oscillator by menas of He's homotopy method and the famous Krylov–Bogoliubov–Mitropolsky method. The approximant allows estimating the points at which the solution crosses the horizontal axis.

1. Introduction

In the standard textbooks, usually the systems with linear damping are considered. Due to their simplicity and the existence of an exact analytical solution, the problem is discussed in details. Unfortunately, in reality, the systems and damping are usually not linear. In recent times, a number of articles have appeared in the literature which deal with the phenomenon of a linear oscillator subject to a quadratic damping force [1–7]. Most elementary textbooks deal with viscous damping for the obvious reason that it involves a linear dependance on the velocity of the oscillator and presents the simplest situation where an exact analytical treatment is possible. In general, this involves the analysis of a second-order ordinary differential equation (ODE) of the Liénard type, namely,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1}$$

Nonlinear equations of motion such as this are seldom addressed in intermediate instruction in classical dynamics; this one is problematic because it cannot be solved in terms of elementary functions. The principal feature associated with quadratic damping is a discontinuous jump of the damping force in the equation of motion whenever the velocity vanishes such that the frictional force always opposes the motion. In this paper, we will consider the following quadratically damped Duffing oscillator ($f(x) = \varepsilon|\dot{x}|$ and $g(x) = \alpha x + \beta x^3$):

$$\ddot{x} + \varepsilon\dot{x}|\dot{x}| + \alpha x + \beta x^3 = 0, \quad x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{2}$$

The quadratically damped oscillator (2) is never critically damped or overdamped, and that to first order in the damping constant, the oscillation frequency is identical to the natural frequency. In the abscence of damping, we obtain the Duffing equation

$$\ddot{x} + \alpha x + \beta x^3 = 0, \quad x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{3}$$

Equation (3) admits the exact analytical solution [8].

$$x(t) = \frac{x_0 cn(\sqrt{\omega}\, t|m) + \dot{x}_0|\sqrt{\omega} sn(\sqrt{\omega}\, t|m) dn(\sqrt{\omega}\, t|m)}{1 + bsn(\sqrt{\omega}\, t|m)^2}, \tag{4}$$

where

$$\omega = \sqrt{\Delta},$$
$$m = \frac{1}{2}\left(1 - \frac{\alpha}{\sqrt{\Delta}}\right),$$
$$b = \frac{1}{2}\left(\frac{\alpha + \beta x_0^2}{\sqrt{\Delta}} - 1\right), \tag{5}$$
$$\Delta = \left(\alpha + \beta x_0^2\right)^2 + 2\beta x_0^2 \neq 0.$$

2. Solution Procedure

In what follows, we will assume that if $x = x(t)$ is a solution to the i.v.p. (2), then

$$\lim_{t \longrightarrow \infty} x(t) = 0. \tag{6}$$

Our aim is to give an approximate analytical solution to the i.v.p. (2). The residual function $R = R(t)$ is defined as follows:

$$R(t) = \ddot{x} + \varepsilon\dot{x}|\dot{x}| + \alpha x + \beta x^3 = \ddot{x} + \delta\varepsilon\dot{x}^2 + \alpha x + \beta x^3, \quad \delta = \pm 1. \tag{7}$$

2.1. First Approach. The ansatz is assumed as

$$x(t) = c_0 e^{-\rho t}\cos\left(\sqrt{\omega}\, t + \cos^{-1}\left(\frac{x_0}{c_0}\right)\right). \tag{8}$$

We have

$$\begin{aligned} R(t) = {} & \frac{1}{4}c_0 \cos(\theta)e^{-\rho t}\left(4\alpha + 3\beta c_0^2 e^{-2\rho t} + 4\rho^2 - 4\omega\right) \\ & + \frac{1}{4}\beta c_0^3 \cos(3\theta)e^{-3\rho t} + \frac{1}{2}c_0^2\delta\varepsilon\cos(2\theta)\left(\rho^2 - \omega\right)e^{-2\rho t} \\ & + c_0^2\delta\varepsilon\rho\sqrt{\omega}\sin(2\theta)e^{-2\rho t} + \frac{1}{2}c_0^2\delta\varepsilon\left(\rho^2 + \omega\right)e^{-2\rho t} + 2c_0\rho\sqrt{\omega}\sin(\theta)e^{-\rho t}, \quad \delta = \pm 1. \end{aligned} \tag{9}$$

Since for small t, $e^{-2\rho t} \approx 1$, we will choose the value of ω so that

$$4\alpha + 3\beta c_0^2 + 4\rho^2 - 4\omega = 0. \tag{10}$$

Then,

$$\omega = \alpha + \frac{3}{4}\beta c_0^2 + \rho^2. \tag{11}$$

The value of c_0 is found from the initial condition $x'(0) = \dot{x}_0$, and it reads

$$c_0 = \pm\sqrt{\frac{3\beta x_0^2 - 4\alpha - 4\rho^2 \pm \sqrt{16\left(\alpha + \rho^2\right)^2 + 3\beta\left(8x_0^2\left(\alpha + 3\rho^2\right) + 3\beta x_0^4 + 32\rho\dot{x}_0 x_0 + 16\dot{x}_0^2\right)}}{6\beta}}. \tag{12}$$

In the case when $\beta \longrightarrow 0$, we define

$$c_0 = \pm\sqrt{\frac{x_0^2\left(\alpha + 2\rho^2\right) + 2\rho\dot{x}_0 x_0 + \dot{x}_0^2}{\alpha + \rho^2}}. \tag{13}$$

The number ρ is a free parameter that is chosen in order to minimize the residual error.

2.2. Second Approach

2.2.1. He's Homotopy Method. We will approximate the expression $\varepsilon\dot{x}|\dot{x}|$ by means of the formula

$$\varepsilon\dot{x}|\dot{x}| \approx r_0\dot{x} + r_1\dot{x}^3 + r_2\dot{x}^5, \quad |\dot{x}| \leq M, \tag{14}$$

where

$$\begin{aligned} r_0 &= -\frac{(\sqrt{3} - 3)M}{3\sqrt{6}}\varepsilon, \\ r_1 &= \frac{\sqrt{2/3}\,(8\sqrt{3} - 9)}{3M}\varepsilon, \\ r_2 &= -\frac{4\sqrt{2/3}\,(2\sqrt{3} - 3)}{3M^3}\varepsilon. \end{aligned} \tag{15}$$

The homotopy equation is defined as

$$H(x, p) = \ddot{x} + \alpha x + p\left[r_0\dot{x} + r_1\dot{x}^3 + r_2\dot{x}^5 + \beta x^3\right]. \tag{16}$$

Following He's approach [9–16], we assume the solution in the ansatz form

$$x(t) = \exp(-\mu p t)\left(r_0 \cos(\omega t + B) + p y_1(\omega t + B)\right), \quad \text{where } \omega = \sqrt{\alpha + p\omega_1}. \tag{17}$$

Let $\tau = \omega t + B$. Then,

$$H(x,p) = \frac{1}{16}\begin{pmatrix} -2\sqrt{\alpha}\, r_0\left(8r_0 + 5\alpha^2 r_0^4 r_2 + 6\alpha r_0^2 r_1 - 16\mu\right)\sin(\tau) + \\ 4r_0\left(3r_0^2\beta - 4\omega_1\right)\cos(\tau) + \\ \left(5\alpha^{5/2} r_0^5 r_2 + 4\alpha^{3/2} r_0^3 r_1\right)\sin(3\tau) + 4r_0^3\beta\cos(3\tau) - \\ \alpha^{5/2} r_0^5 r_2 \sin(5\tau) + 16\alpha y_1(\tau) + 16\alpha y_1''(\tau). \end{pmatrix} p + \cdots. \tag{18}$$

We define

$$\omega_1 = \frac{3r_0^2\beta}{4}, \qquad \mu = \frac{1}{16}\left(8r_0 + 5\alpha^2 r_0^4 r_2 + 6\alpha r_0^2 r_1\right). \tag{19}$$

Then,

$$H(x,p) = \left(5\alpha^{5/2} r_0^5 r_2 + 4\alpha^{3/2} r_0^3 r_1\right)\sin(3\tau) + 4r_0^3\beta\cos(3\tau) - \alpha^{5/2} r_0^5 r_2 \sin(5\tau) + 16\alpha y_1(\tau) + 16\alpha y_1''(\tau). \tag{20}$$

Solving the ode $H(x,p) = 0$ for $y_1 = y_1(\tau)$ gives

$$y_1(\tau) = \frac{12r_0^3\beta\cos(3\tau) + 3\alpha^{3/2} r_0^3\left(5\alpha r_0^2 r_2 + 4r_1\right)\sin(3\tau) - \alpha^{5/2} r_0^5 r_2 \sin(5\tau)}{384\alpha}. \tag{21}$$

The first-order homotopy approximation will then be

$$\left(r_0\cos(w(t)) + \frac{1}{384\alpha}\begin{pmatrix} 12r_0^3\beta\cos(3w(t)) + 3\alpha^{3/2} r_0^3\left(5\alpha r_0^2 r_2 + 4r_1\right) \\ \sin(3w(t)) - \alpha^{5/2} r_0^5 r_2 \sin(5w(t)) \end{pmatrix}\right), \quad \text{where}$$
$$w(t) = \sqrt{\alpha + \frac{3\beta}{4} r_0^2}\, t + B,\ x(t) = \exp\left(-\frac{1}{16}\left(8r_0 + 6\alpha r_1 r_0^2 + 5\alpha^2 r_2 r_0^4\right)t\right). \tag{22}$$

The constants r_0 and r_1 are determined from the initial conditions

$$x(0) = x_0, \qquad x'(0) = \dot{x}_0. \tag{23}$$

2.3. Third Approach

2.3.1. The Krylov–Bogoliubov–Mitropolsky Method (KBM).

The Krylov–Bogoliubov–Mitropolsky Method (KBM) is a technique to give an approximate analytical solution to the weakly nonlinear second-order equation

$$\frac{d^2u}{dt^2} + \omega_0^2 u = \varepsilon f\left(u, \frac{du}{dt}\right). \tag{24}$$

When $\varepsilon = 0$, the solution of (24) may be expressed as

$$u = a\cos(\omega_0 + \theta), \tag{25}$$

where a and θ are constants. For the case when $\varepsilon > 0$ is small, Krylov and Bogoliubov (1947) assumed that the solution is still given by (25) but with time-varying a and θ and subject to the condition

$$\frac{du}{dt} = -a\omega_0 \sin\ \phi, \quad \phi = \omega_0 t + \theta. \tag{26}$$

In the general case, the solution is assumed in the ansatz form

$$u = a\ \cos\ \psi + \sum_{n=1}^{N} \varepsilon^n u_n(a, \psi) + O\left(\varepsilon^{N+1}\right), \tag{27}$$

where each u_n is a periodic function of ψ with a period 2π and a and ψ are assumed to vary with time according to

$$\frac{da}{dt} = \sum_{n=1}^{N} \varepsilon^n A_n(a) + O\left(\varepsilon^{N+1}\right),$$
$$\frac{d\psi}{dt} = \omega_0 + \sum_{n=1}^{N} \varepsilon^n \psi_n(a) + O\left(\varepsilon^{N+1}\right). \tag{28}$$

In order to uniquely determine A_n and ψ_n, we require that no u_n contains cos ψ. Let $N = 3$. Then,

$$\begin{aligned}
\frac{du}{dt} &= -a\omega_0 \sin(\psi) + \left(\omega_0 u_{1,\psi} - a\psi_1 \sin(\psi) + A_1 \cos(\psi)\right)\varepsilon \\
&\quad + \left(A_1 u_{1,a} + \omega_0 u_{2,\psi} + \psi_1 u_{1,\psi} - a\psi_2 \sin(\psi) + A_2 \cos(\psi)\right)\varepsilon^2 \\
&\quad + \left(A_2 u_{1,a} + A_1 u_{2,a} + \omega_0 u_{3,\psi} + \psi_2 u_{1,\psi} + \psi_1 u_{2,\psi} - a\psi_3 \sin(\psi) + A_3 \cos(\psi)\right)\varepsilon^3 + \cdots, \\
\frac{d^2u}{dt^2} &= -a\omega_0^2 \cos(\psi) + \left(\omega_0^2 u_{1,\psi\psi} - 2a\psi_1\omega_0 \cos(\psi) - 2A_1\omega_0 \sin(\psi)\right)\varepsilon \\
&\quad + \begin{pmatrix} 2A_1\omega_0 u_{1,a\psi} + 2\psi_1\omega_0 u_{1,\psi\psi} + \omega_0^2 u_{2,\psi\psi} + \sin(\psi)\left(-aA_1\dot{\psi}_1 - 2A_1\psi_1 - 2A_2\omega_0\right) + \\ \cos(\psi)\left(A_1\dot{A}_1 - a\left(2\psi_2\omega_0 + \psi_1^2\right)\right) \end{pmatrix}\varepsilon^2 \\
&\quad + \begin{pmatrix} \dot{A}_1 A_1 u_{1,a} + A_1^2 u_{1,aa} + 2A_1\psi_1 u_{1,a\psi} + 2A_1\omega_0 u_{2,a\psi} + 2A_2\omega_0 u_{1,a\psi} + \\ A_1\dot{\psi}_1 u_{1,\psi} + 2\psi_2\omega_0 u_{1,\psi\psi} + 2\psi_1\omega_0 u_{2,\psi\psi} + \psi_1^2 u_{1,\psi\psi} + \omega_0^2 u_{3,\psi\psi} + \\ \sin(\psi)\left(-aA_2\dot{\psi}_1 - aA_1\dot{\psi}_2 - 2A_2\psi_1 - 2A_1\psi_2 - 2A_3\omega_0\right) + \\ \cos(\psi)\left(-2a\psi_3\omega_0 - 2a\psi_1\psi_2 + A_2\dot{A}_1 + A_1\dot{A}_2\right) \end{pmatrix}\varepsilon^3 + \cdots.
\end{aligned} \tag{29}$$

Here,

$$\begin{aligned}
\dot{\psi}_1 &= \psi_1'(t), \\
u_{2,\psi} &= \frac{\partial u_2}{\partial \psi}, \\
u_{2,\psi\psi} &= \frac{\partial^2 u_2}{\partial \psi^2}, \\
u_{1,a\psi} &= \frac{\partial^2 u_1}{\partial \psi \partial a}, \text{ etc.}
\end{aligned} \tag{30}$$

Let us consider the i.v.p.

$$\ddot{x} + r_0\dot{x} + r_1\dot{x}^3 + r_2\dot{x}^5 + \alpha x + \beta x^3 = 0, \quad x(0) = x_0, x'(0) = \dot{x}_0 \text{ for } 0 \le t \le T. \tag{31}$$

The values for r_0, r_1, and r_2 are given by (15). Using KBM, we obtain the following odes for determining $a = a(t)$ and $\psi = \psi(t)$:

$$\begin{aligned}\dot{a} = &-\frac{1}{2}\varepsilon s_0 a + \frac{3\varepsilon\left(\beta s_0 - 2\alpha^2 s_1\right)}{16\alpha}a^3 + \frac{\varepsilon\left(-195\beta^2 s_0 - 96\alpha^2\beta s_1 - 99^2\alpha^3\varepsilon^2 s_0 s_1^2 - 320\alpha^4 s_2 + 160^2\alpha^3\varepsilon^2 s_0^2 s_2\right)}{1024\alpha^2}a^5 \\ &- \frac{\varepsilon\left(189\beta^2 s_1 + 117\alpha^3\varepsilon^2 s_1^3 + 720\alpha^2\beta s_2 + 370\alpha^3\varepsilon^2 s_0 s_1 s_2\right)}{8192\alpha}a^7 - \frac{\varepsilon s_2\left(1701\beta^2 + 1782\alpha^3\varepsilon^2 s_1^2 + 2170\alpha^3\varepsilon^2 s_0 s_2\right)}{73728}a^9 \\ &- \frac{29875\alpha^4\varepsilon^3 s_1 s_2^2}{196608}a^{11} - \frac{34625\alpha^5\varepsilon^3 s_2^3}{589824}a^{13}.\end{aligned} \tag{32}$$

$$\begin{aligned}\dot{\psi} = &\frac{8\alpha - \varepsilon^2 s_0^2}{8\sqrt{\alpha}} + \frac{3\beta\left(8\alpha - \varepsilon^2 s_0^2\right)}{64\alpha^{3/2}}a^2 + \frac{-15\beta^2 - 57\alpha\beta\varepsilon^2 s_0 s_1 + 9\alpha^3\varepsilon^2 s_1^2 + 40\alpha^3\varepsilon^2 s_0 s_2}{256\alpha^{3/2}}a^4 \\ &+ \frac{3\left(41\beta^3 - 223\alpha^3\beta\varepsilon^2 s_1^2 - 310\alpha^3\beta\varepsilon^2 s_0 s_2 + 400\alpha^5\varepsilon^2 s_1 s_2\right)}{8192\alpha^{5/2}}a^6 \\ &+ \frac{\alpha^{3/2}\varepsilon^2 s_2\left(-471\beta s_1 + 2200\alpha^2 s_2\right)}{24576}a^8 + \frac{13091\alpha^{5/2}\beta\varepsilon^2 s_2^2}{196608}a^{10}.\end{aligned} \tag{33}$$

$$\begin{aligned}s_0 &= -\frac{(\sqrt{3}-3)M}{3\sqrt{6}}, \\ s_1 &= \frac{\sqrt{2/3}\,(8\sqrt{3}-9)}{3M}, \\ s_2 &= -\frac{4\sqrt{2/3}\,(2\sqrt{3}-3)}{3M^3}, \quad \text{where } M = \max_{0\le t\le T}|\dot{x}(t)|.\end{aligned} \tag{34}$$

The expression for the KBM solution is

$$\begin{aligned}x = x(t) = a\cos(\psi) &+ \frac{a^3\begin{pmatrix} 3072\alpha^2\beta - 2016a^2\alpha\beta^2 + 1251a^4\beta^3 - 576\alpha\beta\varepsilon^2 s_0^2 + 2304\alpha^3\varepsilon^2 s_0 s_1 - \\ 3456a^2\alpha^2\beta\varepsilon^2 s_0 s_1 + 2592a^2\alpha^4\varepsilon^2 s_1^2 - 477a^4\alpha^3\beta\varepsilon^2 s_1^2 - 4756a^4\alpha^3\beta\varepsilon^2 s_0 s_2 + \\ 4080a^4\alpha^5\varepsilon^2 s_1 s_2 + 321a^6\alpha^4\beta\varepsilon^2 s_1 s_2 + 2000a^6\alpha^6\varepsilon^2 s_2^2 + 1300a^8\alpha^5\beta\varepsilon^2 s_2^2 \end{pmatrix}\cos(3\psi)}{98304\alpha^3} \\ &+ \frac{a^3\varepsilon\begin{pmatrix} 884736\alpha\beta s_0 - 1741824a^2\beta^2 s_0 + 1179648\alpha^3 s_1 + 884736a^2\alpha^2\beta s_1 - \\ 943488a^4\alpha\beta^2 s_1 - 221184\alpha^2\varepsilon^2 s_0^2 s_1 + 580608a^2\alpha^3\varepsilon^2 s_0 s_1^2 + \\ 383616a^4\alpha^4\varepsilon^2 s_1^3 + 1474560a^2\alpha^4 s_2 + 1787904a^4\alpha^3\beta s_2 - 257184a^6\alpha^2\beta^2 s_2 - \\ 552960a^2\alpha^3\varepsilon^2 s_0^2 s_2 + 2572800a^4\alpha^4\varepsilon^2 s_0 s_1 s_2 + 1939680a^6\alpha^5\varepsilon^2 s_1^2 s_2 + \\ 2790400a^6\alpha^5\varepsilon^2 s_0 s_2^2 + 2912400a^8\alpha^6\varepsilon^2 s_1 s_2^2 + 1226125a^{10}\alpha^7\varepsilon^2 s_2^3 \end{pmatrix}\sin(3\psi)}{37748736\alpha^{5/2}} \\ &- \frac{a^7\begin{pmatrix} -144\beta^3 + 3024\alpha^3\beta\varepsilon^2 s_1^2 + 2672\alpha^3\beta\varepsilon^2 s_0 s_2 - 3840\alpha^5\varepsilon^2 s_1 s_2 \\ +7548a^2\alpha^4\beta\varepsilon^2 s_1 s_2 - 5200a^2\alpha^6\varepsilon^2 s_2^2 + 285a^4\alpha^5\beta\varepsilon^2 s_2^2 \end{pmatrix}\cos(7\psi)}{4718592\alpha^3}\end{aligned}$$

$$
\begin{aligned}
&-\frac{a^7\varepsilon\begin{pmatrix} -1440\beta^2 s_1 + 1728\alpha^3\varepsilon^2 s_1^3 + 3072\alpha^2\beta s_2 - 1362a^2\alpha\beta^2 s_2 + 9040\alpha^3\varepsilon^2 s_0 s_1 s_2 \\ +16530a^2\alpha^4\varepsilon^2 s_1^2 s_2 + 9200a^2\alpha^4\varepsilon^2 s_0 s_2^2 + 30065a^4\alpha^5\varepsilon^2 s_1 s_2^2 + 15900a^6\alpha^6\varepsilon^2 s_2^3 \end{pmatrix}\sin(7\psi)}{4718592\alpha^{3/2}} \\
&+\frac{\left(a^9\alpha\varepsilon^2 s_2\left(2268\beta s_1 - 400\alpha^2 s_2 + 2477a^2\alpha\beta s_2\right)\right)\cos(9\psi)}{7864320} \\
&+\frac{a^9\varepsilon s_2\begin{pmatrix} -450\beta^2 + 954\alpha^3\varepsilon^2 s_1^2 + 784\alpha^3\varepsilon^2 s_0 s_2 + 3417a^2\alpha^4\varepsilon^2 s_1 s_2 \\ +2580a^4\alpha^5\varepsilon^2 s_2^2 \end{pmatrix}\sin(9\psi)}{4718592\sqrt{\alpha}} \\
&-\frac{\left(1177a^{11}\alpha^2\beta\varepsilon^2 s_2^2\right)\cos(11\psi)}{47185920} - \frac{\left(a^{11}\alpha^{7/2}\varepsilon^3 s_2^2\left(3564s_1 + 4915a^2\alpha s_2\right)\right)\sin(11\psi)}{113246208} \\
&+\frac{\left(1175a^{13}\alpha^{9/2}\varepsilon^3 s_2^3\right)\sin(13\psi)}{792723456}.
\end{aligned}
\tag{35}
$$

The odes (32) and (33) may be solved numerically. However, since we are interested in obtaining analytical solutions, we may limit ourselves to the following approximate solutions:

$$
\begin{aligned}
a &= a(t) = \frac{2A\sqrt{r_0}}{\sqrt{4r_0 e^{r_0 t} + 3\alpha r_1 A^2\left(e^{r_0 t} - 1\right)}}, \\
\psi &= \psi(t) = \sqrt{\alpha}t + B - \frac{\beta}{2\alpha^{3/2} r_1}\left(r_0 t + \log\left(e^{r_0 t} + \frac{3A^2\left(e^{r_0 t} - 1\right)\alpha r_1}{4r_0}\right)\right),
\end{aligned}
\tag{36}
$$

where the values for r_0, r_1, and r_2 are given by (15). The constants A and B are determined from the initial conditions.

2.4. Fourth Approach. We assume the ansatz

$$
x(t) = c_0 e^{-\rho t}\cos\left(f(t) + \cos^{-1}\left(\frac{x_0}{c_0}\right)\right), \quad f(0) = 0. \tag{37}
$$

We have

$$
\begin{aligned}
R(t) &= \frac{1}{4}c_0 \exp(-3\rho t) \\
&\left[\begin{array}{c} \left(4e^{2\rho t}\alpha + 3c_0^2\beta + 4e^{2\rho t}\rho^2 - 4e^{2\rho t}f'(t)^2\right)\cos(\theta) + \\ 2c_0 e^{\rho t}\delta\varepsilon\rho^2 + c_0^2\beta\cos(3\theta) + 2c_0 e^{\rho t}\delta\varepsilon f'(t)^2 + 2c_0 e^{\rho t}\delta\varepsilon\cos(2\theta)\left(\rho^2 - f'(t)^2\right) + \\ 4e^{2\rho t}\left(2\rho f'(t) - f''(t)\right)\sin(\theta) + 4c_0 e^{\rho t}\delta\varepsilon\rho f'(t)\sin(2\theta) \end{array}\right], \\
\delta &= \pm 1.
\end{aligned}
\tag{38}
$$

We will choose the function $f = f(t)$ so that

$$
4e^{2\rho t}\alpha + 3c_0^2\beta + 4e^{2\rho t}\rho^2 - 4e^{2\rho t}f'(t)^2 = 0, \quad f'(t) > 0. \tag{39}
$$

Then,

$$f(t) = F(t) - F(0),$$

$$F(t) = -\frac{\sqrt{4\left(\alpha+\rho^2\right)+3\beta c_0^2 e^{-2\rho t}}}{2\sqrt{\beta}c_0\rho\sqrt{\left(4\left(\alpha+\rho^2\right)e^{2\rho t}/\beta c_0^2\right)+3}} \tag{40}$$

$$\left(\sqrt{\beta}\,c_0\sqrt{\frac{4\left(\alpha+\rho^2\right)e^{2\rho t}}{\beta c_0^2}+3} - 2\sqrt{\alpha+\rho^2}\,e^{\rho t}\sinh^{-1}\left(\frac{2\sqrt{\alpha+\rho^2}\,e^{\rho t}}{\sqrt{3}\,\sqrt{\beta}\,c_0}\right)\right).$$

The value of c_0 is found from the initial condition $x'(0) = \dot{x}_0$, and it reads

$$c_0 = \pm\sqrt{\frac{3\beta x_0^2 - 4\alpha - 4\rho^2 \pm \sqrt{16\rho^4 + 8\rho^2\left(4\alpha+9\beta x_0^2\right)+\left(4\alpha+3\beta x_0^2\right)^2 + 96\beta\rho x_0\dot{x}_0 + 48\beta\dot{x}_0^2}}{6\beta}}. \tag{41}$$

In the case when $\beta \longrightarrow 0$, we define

$$c_0 = \pm\sqrt{\frac{1}{2}\left(x_0^2 + \frac{x_0^2\left(\alpha+3\rho^2\right)+4\rho\dot{x}_0x_0+2\dot{x}_0^2}{\left|\alpha+\rho^2\right|}\right)}. \tag{42}$$

3. Analysis and Discussion

In this section, we will compare the accuracy of the solution methods using the different approaches described in the previous section.

Example 1. Let us consider the i.v.p.

$$\ddot{x} + 0.2\dot{x}|\dot{x}| + x + 2x^3 = 0, \quad x(0) = 0,\ x'(0) = 0.1. \tag{43}$$

The approximate analytical solution using the first approach (see formula (8)) for $\rho = 0.007191$ is

$$x_{\text{approx}}(t) = 0.0992665e^{-0.007191t}\ \sin(1.00739t). \tag{44}$$

It is shown in Figure 1.

The solution obtained by means of He's homotopy method (see Figure 2) equals

$$x_{\text{He}}(t) = e^{-0.0089515t}\begin{pmatrix} -0.0000427888\ \sin(3.00025(t+7.85294)) - 3.49e- \\ 6\ \sin(5.00041(t+7.85294)) - 0.0999944\ \cos(1.00008(t+7.85294)) \\ -1.03e-6\ \cos(3.00025(t+7.85294)) - 6.9e-8\ \cos(5.00041(t+7.85294)) \end{pmatrix}. \tag{45}$$

Using the KBM, we obtained the following solution (Figure 3):

$$
\begin{aligned}
&6.736e-9\,\sin\left(\begin{array}{c}-1.89144\,\log\left(0.850343-1.e^{0.00345092t}\right)+\\ (4.2634+5.94212i)-4.99347t\end{array}\right)+\left(1.32e-6-1.43e-6e^{0.00345092t}\right)\\
&-\sin\left(-1.13486\,\log\left(0.850343-1.e^{0.00345092t}\right)+(2.55804+3.56527i)-2.99608t\right)\\
&\left(1.1e-6e^{0.00345092t}-9.2e-7\right)\cos\left(\begin{array}{c}-1.13486\,\log\left(0.850343-1.e^{0.00345092t}\right)+\\ (2.55804+3.56527i)-2.99608t\end{array}\right)+\\
&\left(0.00844854-0.0198709e^{0.00345092t}+0.0116841e^{0.00690184t}\right),\\
x_{\mathrm{KBM}}(t)=&\frac{\cos\left(-0.378287\,\log\left(0.850343-1.e^{0.00345092t}\right)+(0.85268+1.18842i)-0.998695t\right)}{\sqrt{0.0922354e^{0.00345092t}-0.0784317\left(0.850343-e^{0.00345092t}\right)^2}}.
\end{aligned}
\tag{46}
$$

Now, using the fourth approach, we get the solution (see Figure 4)

$$
\begin{aligned}
x_{\mathrm{trig}}(t)=&-0.0992665e^{-0.00719t}\\
&\sin\left(\begin{array}{c}69.541\sqrt{4.00021+0.0591231e^{-0.01438t}}-\\ \dfrac{4.8811e^{0.00719t}\sqrt{202.977e^{0.01438t}+3}\,\sqrt{4.00021+0.0591231e^{-0.01438t}}\,\sinh^{-1}\left(8.22551e^{0.00719t}\right)}{0.01478+1.e^{0.01438t}}+249.895\end{array}\right).
\end{aligned}
\tag{47}
$$

Example 2. Let us consider the i.v.p.

$$\ddot{x}+0.2\dot{x}|\dot{x}|+x+10x^3=0,\quad x(0)=0,\ x'(0)=0.1. \tag{48}$$

The approximate analytical solution for $\rho=0.0084$ using the fourth approach is

$$
\begin{aligned}
x_{\mathrm{approx}}(t)=&\\
&0.097e^{-0.0084t}\sin\left(\begin{array}{c}59.5231\\ \dfrac{}{2.067-\sqrt{4.0003+0.280334e^{-0.017t}}-}\\ 0.611395\left(\dfrac{6.672-}{\dfrac{e^{0.0084t}\sqrt{42.81e^{0.017t}+3}\,\sqrt{4.0003+0.28e^{-0.017t}}\,\sinh^{-1}\left(3.778e^{0.0084t}\right)}{0.28+4.0003e^{0.017t}}}\right)\end{array}\right).
\end{aligned}
\tag{49}
$$

It is shown in Figure 5.

The obtained results may be applied to solve the pendulum equation with quadratic damping

$$\ddot{x}+\varepsilon\dot{x}|\dot{x}|+\omega^2\sin x=0,\quad x(0)=x_0,\ x'(0)=\dot{x}_0. \tag{50}$$

Indeed, we may use the approximation

$$\sin x\approx x-\frac{3}{19}x^3 \text{ for } |x|\le\frac{\pi}{3}, \tag{51}$$

and then, we replace i.v.p. (43) with the i.v.p.

$$\ddot{x}+\varepsilon\dot{x}|\dot{x}|+\omega^2x-\frac{3\omega^2}{19}x^3=0,\quad x(0)=x_0,\ x'(0)=\dot{x}_0. \tag{52}$$

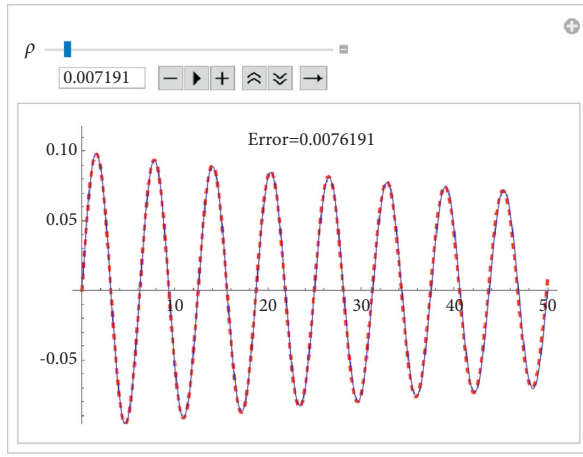

Figure 1: Comparison between the analytical approximant (44) (formula (8)) and the Runge–Kutta numerical solution for the i.v.p (43).

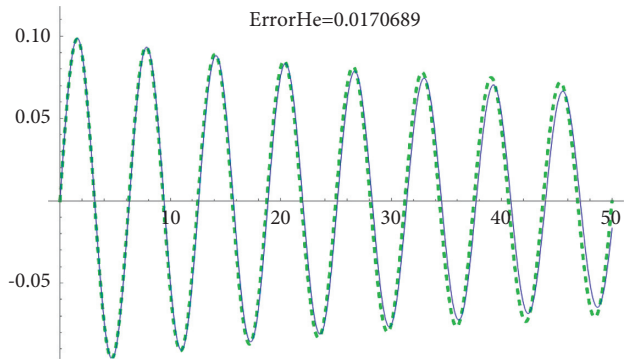

Figure 2: He's homotopy method-error comparison with the Runge–Kutta numerical solution.

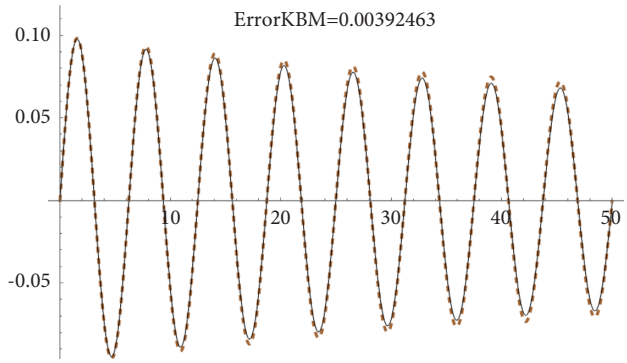

Figure 3: KBM-error comparison with the Runge–Kutta numerical solution.

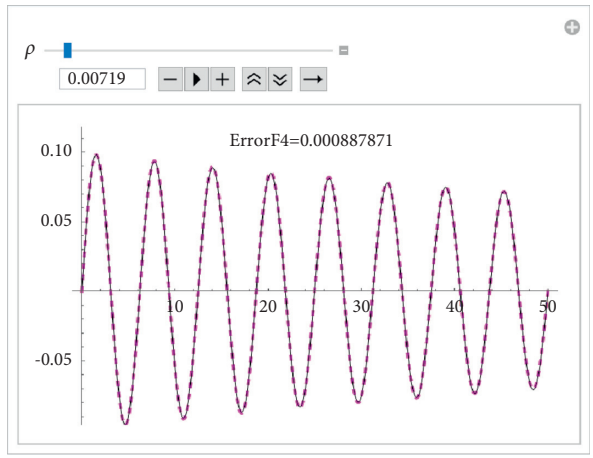

Figure 4: Method using the fourth approach-error comparison with the Runge–Kutta numerical solution.

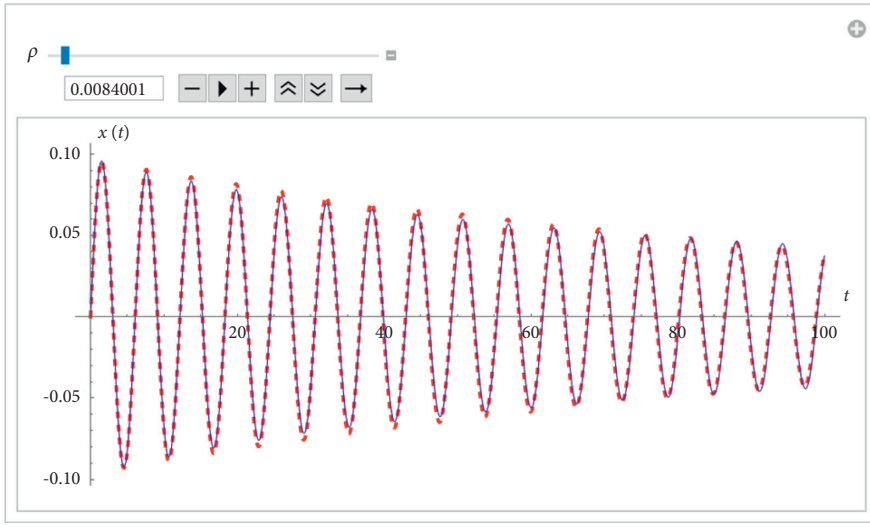

Figure 5: Comparison between the analytical approximant (49) and the Runge–Kutta numerical solution for the i.v.p (48).

4. Conclusions

We have obtained approximate analytical solutions to the quadratically damped Duffing oscillator equation by means of an elementary approach. We introduced a ρ−parameter technique that allowed us to optimize the obtained solution. The results are also valid for the linear quadratically damped oscillator $\ddot{x} + \varepsilon\dot{x}|\dot{x}| + \alpha x = 0$. A similar approach may be employed to study the quadratically damped cubic-quintic oscillator $\ddot{x} + \varepsilon\dot{x}|\dot{x}| + \alpha x + \beta x^3 + \gamma x^5 = 0$. Also, a more general quadratically damped oscillator $\ddot{x} + \varepsilon\dot{x}|\dot{x}| + h(x) = 0$ may be solved for any odd parity function $h(x)$. In future work, we will study quadratically damped forced oscillators having the form $\ddot{x} + \varepsilon\dot{x}|\dot{x}| + h(x) = F(t)$ for any continuous functions $h(x)$ and $F(t)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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