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Research Article

Analytical Approximant to a Quadratically Damped Forced Cubic-Quintic Duffing Oscillator

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The cubic-quintic Duffing oscillator of a system with strong quadratic damping and forcing is considered. We give elementary approximate analytical solution to this oscillator in terms of exponential and trigonometric functions. We compare the analytical approximant with the Runge–Kutta numerical solution. The approximant allows us to estimate the points at which the solution crosses the horizontal axis.

1. Introduction

In this paper, some novel analytical and numerical techniques are introduced for analyzing and solving nonlinear ordinary differential equations (NODEs) that are associated to some strongly nonlinear oscillators such as a quadratically damped cubic-quintic Duffing equation. There are many numerical and analytical approaches that were applied for solving the second-order nonlinear oscillator equations. For instance, both the homotopy perturbation method (HBM) and MTS technique were applied for analyzing a forced Van der Pol (VdP) generalized oscillator to obtain the amplitudes of the forced harmonic and super and subharmonic oscillatory states[1]. Also, Melnikov's method was employed for analyzing a mVdPD equation and deriving analytical criteria for the appearance of horseshoe chaos in chemical oscillations [2]. He et al. [3] used the Poincare–Lindstedt technique (PLT) for solving and analyzing the hybrid Rayleigh–Van der Pol–Duffing equation. Moreover, the homotopy analysis method (HAM) was employed for analyzing the DVdP oscillator [4]. Both methods of differentiable dynamics and Lie symmetry reduction method were devoted for analyzing the DVdP-type oscillator [5]. The principal feature associated with quadratic damping is a discontinuous jump of the damping force in the equation of motion whenever the velocity vanishes such that the frictional force always opposes the motion.

In this paper, we will consider the following quadratically damped and forced cubic-quintic Duffing oscillator:

$$\ddot{x} + \varepsilon\dot{x}|\dot{x}| + \alpha x + \beta x^3 + \gamma x^5 = F(t), \quad x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{1}$$

The quadratically damped oscillator (1) is never critically damped or overdamped. In the absence of damping and forcing, we obtain the cubic-quintic Duffing equation

$$\ddot{x} + \alpha x + \beta x^3 + \gamma x^5 = 0, \quad x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{2}$$

Equation (2) admits exact analytical solution that is expressed in terms of the Jacobian elliptic functions. Other solution methods may be found in [1–8].

2. Solution Procedure

2.1. First Case: Undamped and Unforced Cubic-Quintic Duffing Equation. Let us consider the i.v.p.

$$\ddot{x} + \alpha x + \beta x^3 + \gamma x^5 = 0, \quad x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{3}$$

Assume the ansatz

$$x(t) = \frac{v(t)}{\sqrt{1 + \lambda v^2(t) + \mu v^4(t)}}, \tag{4}$$

where the function $v = v(t)$ is the solution to some Duffing equation

$$\ddot{v} + pv + qv^3 = 0, \quad v(0) = v_0 \text{ and } v'(0) = \dot{v}_0. \tag{5}$$

The numbers v_0 and $\dot{v}_0$ are determined from the initial conditions. Observe that

$$\dot{v}_0 = -\frac{\dot{x}_0\left(\lambda v_0^2 + \mu v_0^4 + 1\right)^{3/2}}{\mu v_0^4 - 1}. \tag{6}$$

From (5), it follows that

$$\dot{v}^2 = \dot{v}_0^2 + pv_0^2 + \frac{qv_0^4}{2} - pv^2 - \frac{qv^4}{2}. \tag{7}$$

Observe that

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4 + \frac{1}{6}\gamma x^6 - \frac{\alpha x_0^2}{2} - \frac{\beta x_0^4}{4} - \frac{1}{6}\gamma x_0^6 - \frac{\dot{x}_0^2}{2} = 0. \tag{8}$$

We have

$$\begin{aligned}0 &= \frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4 + \frac{1}{6}\gamma x^6 - \frac{\alpha x_0^2}{2} - \frac{\beta x_0^4}{4} - \frac{1}{6}\gamma x_0^6 - \frac{\dot{x}_0^2}{2} \\ &= \frac{1}{12\left(1 + \lambda v^2 + \mu v^4\right)} \sum_{j=0}^{6} S_j v^{2j},\end{aligned} \tag{9}$$

where

$$\begin{aligned}
S_0 &= 6pv_0^2 + 3qv_0^4 + 6\dot{v}_0^2 - 6\alpha x_0^2 - 3\beta x_0^4 - 2\gamma x_0^6 - 6\dot{x}_0^2, \\
S_1 &= -3\left(-2\alpha + 2p + 6\alpha\lambda x_0^2 + 3\beta\lambda x_0^4 + 2\gamma\lambda x_0^6 + 6\lambda\dot{x}_0^2\right), \\
S_2 &= -3\left(-4\alpha\lambda - \beta + 4\mu pv_0^2 + 2\mu qv_0^4 + q + 4\mu\dot{v}_0^2 + 6\alpha\lambda^2 x_0^2\right. \\
&\quad + 6\alpha\mu x_0^2 + 3\beta\lambda^2 x_0^4 + 3\beta\mu x_0^4 + 2\gamma\lambda^2 x_0^6 + 2\gamma\mu x_0^6 \\
&\quad \left. + 6\lambda^2\dot{x}_0^2 + 6\mu\dot{x}_0^2\right), \\
S_3 &= 6\alpha\lambda^2 + 12\alpha\mu + 3\beta\lambda + 2\gamma + 12\mu p - 6\alpha\lambda^3 x_0^2 - 36\alpha\lambda\mu x_0^2 \\
&\quad - 3\beta\lambda^3 x_0^4 - 18\beta\lambda\mu x_0^4 - 2\gamma\lambda^3 x_0^6 - 12\gamma\lambda\mu x_0^6 - 6\lambda^3\dot{x}_0^2 \\
&\quad - 36\lambda\mu\dot{x}_0^2, S_4 = 3\mu\left(4\alpha\lambda + \beta + 2\mu pv_0^2 + \mu qv_0^4 + 2q + 2\mu\dot{v}_0^2\right. \\
&\quad - 6\alpha\lambda^2 x_0^2 - 6\alpha\mu x_0^2 - 3\beta\lambda^2 x_0^4 - 3\beta\mu x_0^4 - 2\gamma\lambda^2 x_0^6 - 2\gamma\mu x_0^6 \\
&\quad \left. - 6\lambda^2\dot{x}_0^2 - 6\mu\dot{x}_0^2\right), \\
S_5 &= -3\mu^2\left(-2\alpha + 2p + 6\alpha\lambda x_0^2 + 3\beta\lambda x_0^4 + 2\gamma\lambda x_0^6 + 6\lambda\dot{x}_0^2\right), \\
S_6 &= -\mu^2\left(3q + 6\alpha\mu x_0^2 + 3\beta\mu x_0^4 + 2\gamma\mu x_0^6 + 6\mu\dot{x}_0^2\right).
\end{aligned} \tag{10}$$

Eliminating p, q, μ, v_0, and $\dot{v}_0$ from the system $S_0 = S_1 = S_2 = S_3 = S_4 = S_5 = S_6 = 0$ gives

$$\begin{aligned}&\left(\left(6\alpha x_0^2 + 3\beta x_0^4 + 2\gamma x_0^6 + 6\dot{x}_0^2\right)\lambda^3 - 6\alpha\lambda^2 - 3\beta\lambda - 2\gamma\right) \\ &\left(\begin{array}{c} 4\left(6\alpha x_0^2 + 3\beta x_0^4 + 2\gamma x_0^6 + 6\dot{x}_0^2\right)^2\lambda^3 - 24\alpha\left(6\alpha x_0^2 + 3\beta x_0^4 + 2\gamma x_0^6 + 6\dot{x}_0^2\right)\lambda^2 \\ +3\left(12\alpha^2 - 6\alpha\beta x_0^2 - 3\beta^2 x_0^4 - 2\beta\gamma x_0^6 - 6\beta\dot{x}_0^2\right)\lambda + 9\alpha\beta + 6\alpha\gamma x_0^2 + 3\beta\gamma x_0^4 + 2\gamma^2 x_0^6 + 6\gamma\dot{x}_0^2 \end{array}\right) = 0.\end{aligned} \tag{11}$$

The values for p, q, and μ are

$$\begin{aligned}
p &= \frac{1}{2}\left(2\alpha - 6\alpha\lambda x_0^2 - 3\beta\lambda x_0^4 - 2\gamma\lambda x_0^6 - 6\lambda\dot{x}_0^2\right), \\
q &= \frac{1}{3}\left(12\alpha\lambda + 3\beta - 18\alpha\lambda^2 x_0^2 - 30\alpha\mu x_0^2 - 9\beta\lambda^2 x_0^4\right. \\
&\quad \left. - 15\beta\mu x_0^4 - 6\gamma\lambda^2 x_0^6 - 10\gamma\mu x_0^6 - 18\lambda^2\dot{x}_0^2 - 30\mu\dot{x}_0^2\right), \\
\mu &= \frac{6\alpha\lambda^2 + 3\beta\lambda + 2\gamma - 6\alpha\lambda^3 x_0^2 - 3\beta\lambda^3 x_0^4 - 2\gamma\lambda^3 x_0^6 - 6\lambda^3\dot{x}_0^2}{12\left(-2\alpha + 6\alpha\lambda x_0^2 + 3\beta\lambda x_0^4 + 2\gamma\lambda x_0^6 + 6\lambda\dot{x}_0^2\right)}.
\end{aligned} \tag{12}$$

The value of v_0 is found from the condition $x(0) = x_0$:

$$v_0 = \pm\sqrt{\frac{1 - \lambda x_0^2 \pm \sqrt{\left(\lambda x_0^2 - 1\right)^2 - 4\mu x_0^4}}{2\mu x_0^2}}. \tag{13}$$

On the other hand, the solution to i.v.p. (5) is given by

$$v(t) = c\,\mathrm{cn}\left(\sqrt{qc^2 + p}\,t + cn^{-1}\left(\frac{v_0}{c}, \frac{c^2 q}{2\left(qc^2 + p\right)}\right), \frac{c^2 q}{2\left(qc^2 + p\right)}\right), \tag{14}$$

where

$$c = \pm\sqrt{\frac{-p \pm \sqrt{\left(p + qv_0^2\right)^2 + 2q\dot{v}_0^2}}{q}}. \tag{15}$$

The number

$$\Delta = \left(p + qv_0^2\right)^2 + 2q\dot{v}_0^2, \tag{16}$$

is called the discriminant to Duffing equation (5). In the case when $\Delta > 0$, this solution may also be written in the form

$$v(t) = \frac{v_0 cn(t\sqrt{\omega} \mid m) + (\dot{v}_0/\sqrt{\omega}) dn(t\sqrt{\omega} \mid m) sn(t\sqrt{\omega} \mid m)}{1 + \left(\left(p + qv_0^2 - \omega\right)/2\omega\right) sn(t\sqrt{\omega} \mid m)^2}, \tag{17}$$

where

$$\omega = -\frac{p}{2m-1},$$
$$m = \frac{1}{2}\left(1 \pm \frac{p}{\sqrt{\left(p+qv_0^2\right)^2 + 2q\dot{v}_0^2}}\right). \tag{18}$$

Suppose that $\Delta < 0$. Define

$$\delta = 2pv_0^2 + qv_0^4 + 2\dot{v}_0^2. \tag{19}$$

Since $\Delta < 0$, necessarily $q < 0$. From the equality

$$\delta = \frac{p^2 - \Delta}{-q}, \tag{20}$$

it is evident that $\delta > 0$.Then, the solution to i.v.p. (5) reads

$$v(t) = \sqrt[4]{\frac{\delta}{-q}} - \frac{2\sqrt[4]{\delta/-q}}{\left(1 + u_0 cn(\sqrt{\overline{\omega}}\,x|\overline{m}) + (\dot{u}_0/\sqrt{\overline{\omega}})sn(\sqrt{\overline{\omega}}\,x|\overline{m})dn(\sqrt{\overline{\omega}}\,x|\overline{m})\right)/\left(1 + \overline{b}sn(\sqrt{\overline{\omega}}\,x|\overline{m})^2\right)}, \tag{21}$$

where

$$\overline{\omega} = \sqrt{\overline{\Delta}},$$
$$\overline{m} = \frac{1}{2} \pm \frac{\overline{\alpha}}{2\sqrt{\overline{\Delta}}},$$
$$\overline{b} = \frac{\overline{\alpha} + \overline{\beta}u_0^2}{2\sqrt{\overline{\Delta}}} - \frac{1}{2},$$
$$u_0 = \frac{A + v_0}{A - v_0} \text{ and } \dot{u}_0 = \frac{2A\dot{v}_0^2}{(A - v_0)^2},$$
$$\overline{\Delta} = \left(\overline{\alpha} + u_0^2\overline{\beta}\right)^2 + 2\overline{\beta}\dot{u}_0^2, \tag{22}$$
$$\overline{\alpha} = \frac{1}{2}\left(3A^2q - p\right),$$
$$\overline{\beta} = \frac{1}{2}\left(A^2q + p\right),$$
$$A = \sqrt[4]{\frac{\delta}{-q}},$$
$$\delta = 2pv_0^2 + qv_0^4 + 2\dot{v}_0^2.$$

Observe that

$$\overline{\Delta} = \left(\overline{\alpha} + u_0^2\overline{\beta}\right)^2 + 2\overline{\beta}\dot{u}_0^2 = 2\left(\sqrt{p^2(-q)\delta} + (-q)\delta\right) > 0. \tag{23}$$

Example 1. Let us consider the i.v.p.

$$\ddot{x} - 1.25819x - 2.40487x^3 + 0.299493x^5 = 0,$$
$$x(0) = -0.261431 \text{ and } x'(0) = 0.770472. \tag{24}$$

The exact solution is given by

$$x_{\text{exact}}(t) = \frac{v(t)}{\sqrt{1 - 1.2914v^2(t) + 0.231351v^4(t)}}, \tag{25}$$

where

$$v(t) = \frac{-0.250491cn(0.916322t|0.0909919) + 0.738957\ dn(0.916322t|0.0909919)sn(0.916322t|0.0909919)}{1 - 0.0823122sn^2(0.916322t|0.0909919)}. \tag{26}$$

See Figure 1.

2.2. Solution by Means of He's Frequency Method. Let

$$f(x) = \alpha x + \beta x^3 + \gamma x^5. \tag{27}$$

He's method assumes the solution in the ansatz form

$$x(t) = A\cos\left(\sqrt{\omega}t + \cos^{-1}\left(\frac{x_0}{A}\right)\right), \quad A \neq 0. \tag{28}$$

The frequency is evaluated by means of the formula

$$\omega = \frac{f(x)}{x} \quad \text{at} \quad x = \frac{\sqrt{3}}{2}A. \tag{29}$$

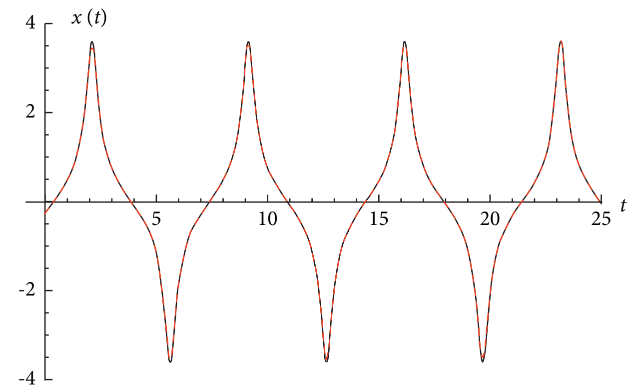

Figure 1: Comparison between the exact solution and the Runge–Kutta numerical solution for i.v.p. (24).

Then,

$$\omega = \omega_{He} = \alpha + \frac{3\beta}{4}A^2 + \frac{9\gamma}{16}A^4. \tag{30}$$

The number A is found from the initial condition $x'(0) = \dot{x}_0$ so that

$$A = \pm\sqrt{\frac{\dot{x}_0^2}{\alpha + \left(9A^4\ \gamma/16\right) + \left(3A^2\beta/4\right)} + x_0^2}. \tag{31}$$

2.3. Solution by Means of a Simple Trigonometric Ansatz. As in He's approach, we assume the solution in ansatz form (28) so that

$$\begin{aligned}\ddot{x} + \alpha x + \beta x^3 + \gamma x^5 &= \frac{1}{16}A^5\gamma\ \cos(5\theta)\\ &= +\frac{1}{16}\cos(3\theta)\left(5A^5\gamma + 4A^3\beta\right)\\ &\quad + \frac{1}{8}A\ \cos(\theta)\left(8\alpha + 5A^4\gamma + 6A^2\beta - 8\omega\right).\end{aligned} \tag{32}$$

We choose the frequency ω so that

$$8\alpha + 5A^4\gamma + 6A^2\beta - 8\omega = 0. \tag{33}$$

Then,

$$\omega = \omega_{\text{trigo}} = \alpha + \frac{3\beta}{4}A^2 + \frac{10\gamma}{16}A^4. \tag{34}$$

This last formula looks like He's formula (30). The difference is $\omega_{\text{trigo}} - \omega_{\text{He}} = (\gamma/16)A^4$. This suggests to consider the following κ-parameter solution:

$$x(\kappa,t) = x(t) = A\cos\left(\sqrt{\alpha + \frac{3\beta}{4}A^2 + \frac{9\gamma}{16}\kappa A^4}\,t + \cos^{-1}\left(\frac{x_0}{A}\right)\right), \tag{35}$$

where the number A is a solution to the sextic

$$\frac{1}{16}\gamma\lambda A^6 + \left(\frac{3\beta}{4} - \frac{1}{16}\gamma\lambda x_0^2\right)A^4 + \left(\alpha - \frac{3\beta x_0^2}{4}\right)A^2 - \alpha x_0^2 - \dot{x}_0^2 = 0. \tag{36}$$

The number λ is chosen in order to get as small residual error as possible.

2.4. Solution by Means of an Improved Trigonometric Ansatz

2.4.1. First Improved Ansatz. Let us consider the i.v.p.

$$\ddot{x} + \alpha x + \beta x^3 + \gamma x^5 = 0, \quad x(0) = A \text{ and } x'(0) = 0. \tag{37}$$

Assume the ansatz

$$x(t) = A\frac{\sqrt{1+\lambda+\mu}\cos(\sqrt{\omega}t)}{\sqrt{1 + \lambda\ \cos^2(\sqrt{\omega}t) + \mu\ \cos^4(\sqrt{\omega}t)}}. \tag{38}$$

Let

$$R(t) = x''(t) + \alpha x(t) + \beta x^3(t) + \gamma x^5(t). \tag{39}$$

The numbers λ, μ, and ω are found from the conditions

$$\begin{aligned} &R(0) = R''(0) = R''''(0) = 0,\\ &\lambda = -\frac{2\left(4\alpha^2 - \alpha\omega + 2A^8\gamma^2 + 5A^6\ \beta\gamma + 6\alpha A^4\gamma + 3A^4\beta^2 - A^4\gamma\omega + 7\alpha A^2\beta - A^2\beta\omega - 3\omega^2\right)}{\left(\alpha + A^4\gamma + A^2\beta\right)\left(4\alpha + 2A^4\gamma + 3A^2\beta + 2\omega\right)},\\ &\mu = \frac{4\alpha + 2A^4\gamma + 3A^2\beta - 4\omega}{4\alpha + 2A^4\gamma + 3A^2\beta + 2\omega}.\end{aligned} \tag{40}$$

The frequency ω is found from the quadratic equation

$$\begin{aligned}&34\alpha^2 + A^4\left(2A^2\gamma + 3\beta\right)\left(5A^2\gamma + 6\beta\right) + \alpha\left(40A^4\gamma + 51A^2\beta\right)\\ &\quad - 5\left(4\alpha + 2A^4\gamma + 3A^2\beta\right)\omega - 14\omega^2 = 0.\end{aligned} \tag{41}$$

2.4.2. Second Improved Ansatz. Let us consider the i.v.p.

$$\ddot{x} + \alpha x + \beta x^3 + \gamma x^5 = 0, \quad x(0) = A \text{ and } x'(0) = 0. \tag{42}$$

Assume the ansatz

$$x(t)=A\frac{\sqrt{1+\lambda+\mu+\upsilon}\cos(\sqrt{\omega t})}{\sqrt{1+\lambda\cos^2(\sqrt{\omega t})+\mu\cos^4(\sqrt{\omega t})+\upsilon\cos^6(\sqrt{\omega t})}}. \tag{43}$$

The numbers λ, μ, υ, and ω are found from the conditions

$$R(0)=R''(0)=R^{(4)}(0)=R^{(6)}(0)=0,$$

$$\lambda=-\frac{3\begin{pmatrix}34\alpha^3+20\alpha^2\omega-24\alpha\omega^2+10A^{12}\gamma^3+37A^{10}\beta\gamma^2+\\ 50\alpha A^8\gamma^2+45A^8\beta^2\gamma+10A^8\gamma^2\omega+118\alpha A^6\beta\gamma+\\ 18A^6\beta^3+25A^6\beta\gamma\omega+74\alpha^2A^4\gamma+69\alpha A^4\beta^2+30\alpha A^4\gamma\omega+\\ 15A^4\beta^2\omega-24A^4\gamma\omega^2+85\alpha^2A^2\beta+35\alpha A^2\beta\omega-24A^2\beta\omega^2-30\omega^3\end{pmatrix}}{\left(\alpha+A^4\gamma+A^2\beta\right)\begin{pmatrix}34\alpha^2+40\alpha\omega+10A^8\gamma^2+27A^6\beta\gamma+\\ 40\alpha A^4\gamma+18A^4\beta^2+20A^4\gamma\omega+51\alpha A^2\beta+30A^2\beta\omega+16\omega^2\end{pmatrix}}, \tag{44}$$

$$\mu=\frac{3\left(34\alpha^2+10A^8\gamma^2+27A^6\beta\gamma+40\alpha A^4\gamma+18A^4\beta^2+51\alpha A^2\beta-34\omega^2\right)}{34\alpha^2+40\alpha\omega+10A^8\gamma^2+27A^6\beta\gamma+40\alpha A^4\gamma+18A^4\beta^2+20A^4\gamma\omega+51\alpha A^2\beta+30A^2\beta\omega+16\omega^2},$$

$$\upsilon=-\frac{34\alpha^2-20\alpha\omega+10A^8\gamma^2+27A^6\beta\gamma+40\alpha A^4\gamma+18A^4\beta^2-10A^4\gamma\omega+51\alpha A^2\beta-15A^2\beta\omega-14\omega^2}{34\alpha^2+40\alpha\omega+10A^8\gamma^2+27A^6\beta\gamma+40\alpha A^4\gamma+18A^4\beta^2+20A^4\gamma\omega+51\alpha A^2\beta+30A^2\beta\omega+16\omega^2}.$$

The frequency ω is found from the cubic equation

$$\begin{aligned}&-\left(4\alpha+3A^2\beta+2A^4\gamma\right)\left(124\alpha^2+186A^2\alpha\beta+63A^4\beta^2\right.\\&\left.+160A^4\alpha\gamma+102A^6\beta\gamma+40A^8\gamma^2\right)\\&+84\left(4\alpha+3A^2\beta+2A^4\gamma\right)\omega+160\omega^3=0.\end{aligned} \tag{45}$$

2.5. Homotopy Method. Consider the homotopy

$$H(x,p)=\ddot{x}+\alpha x+p\left(\beta x^3+\gamma x^5\right), \tag{46}$$

and assume the solution in then ansatz form

$$\begin{aligned}x(t)=&\,y_0(\omega t)+py_1(\omega t)+p^2y_2(\omega t)+p^3y_3(\omega t)+p^4y_4(\omega t)\\&+p^5y_5(\omega t)+\cdots,\end{aligned} \tag{47}$$

where $\omega=\sqrt{\alpha+p\omega_1+p^2\omega_2+p^3\omega_2+p^4\omega_2+p^5\omega_2+\cdots}$.

Plugging the expression for $x(t)$ into $H(x,p)$ and equating the coefficients of p^j $(j=0,1,2,3,\ldots)$ will give an ode system. Solving this system so that no secular terms appear will give the following expressions: y.

2.6. Second Case: Quadratically Damped and Unforced Oscillator. Our aim is to give *approximate* analytical solution to i.v.p. (1). Define the residual function $R=R(t)$ as follows:

$$R(t)=\ddot{x}+\varepsilon\dot{x}|\dot{x}|+\alpha x+\beta x^3+\gamma x^5=\ddot{x}\pm\varepsilon\dot{x}^2+\alpha x+\beta x^3+\gamma x^5. \tag{48}$$

2.6.1. First Approach. Assume the ansatz

$$x(t)=c_0e^{-\rho t}\cos\left(f(t)+\cos^{-1}\left(\frac{x_0}{c_0}\right)\right). \tag{49}$$

Let $\theta=f(t)+\cos^{-1}(x_0/c_0)$. We have

$$\begin{aligned}R(t)=&\,c_0\left(8e^{4t\rho}\alpha+8e^{4t\rho}\rho^2+6e^{2t\rho}\beta c_0^2+5\gamma c_0^4-8e^{4t\rho}f'(t)^2\right)\\&\cos(\theta)\\&+\frac{1}{16}e^{-5t\rho}\gamma\cos(5\theta)f'(t)c_0^5+\frac{1}{16}e^{-5t\rho}c_0^3\left(4e^{2t\rho}\beta+5\gamma c_0^2\right)\\&\cos(3\theta)\\&+e^{-2t\rho}\varepsilon\rho c_0^2f'(t)\sin(2\theta)+\frac{1}{2}e^{-2t\rho}\varepsilon c_0^2\left(\rho^2-f'(t)^2\right)\\&\cos(2\theta)\\&+\frac{1}{2}e^{-2t\rho}\varepsilon c_0^2\left(\rho^2+f'(t)^2\right)+\frac{1}{8}e^{-5t\rho}+e^{-t\rho}c_0(2\rho f'(t)\\&-f''(t))\sin(\theta).\end{aligned} \tag{50}$$

We will choose the function $f = f(t)$ so that

$$8e^{4t\rho}\alpha + 8e^{4t\rho}\rho^2 - 8e^{4t\rho}f'(t)^2 + 6e^{2t\rho}\beta c_0^2 + 5\gamma c_0^4 = 0, \text{ and } f(0) = 0 \text{ and } f'(t) > 0. \tag{51}$$

Then,

$$f(t) = \frac{1}{2\sqrt{2}}\int_0^t \sqrt{8\alpha + 8\rho^2 + 6e^{-2\rho\tau}\beta c_0^2 + 5e^{-4\rho\tau}\gamma c_0^4}\,d\tau = F(t) - F(0), \tag{52}$$

where

$$F(t) = \frac{\sqrt{8(\alpha+\rho^2) + 6\beta c_0^2 e^{-2\rho t} + 5\gamma c_0^4 e^{-4\rho t}}}{40\sqrt{\gamma}\rho\sqrt{5\gamma c_0^4 + 6\beta c_0^2 e^{2\rho t} + 8(\alpha+\rho^2)e^{4\rho t}}}, \left(-5\sqrt{2}\sqrt{\gamma}\sqrt{5\gamma c_0^4 + 6\beta c_0^2 e^{2\rho t} + 8(\alpha+\rho^2)e^{4\rho t}} + 20\sqrt{\gamma}\sqrt{\alpha+\rho^2}\,e^{2\rho t} \tanh^{-1}\left(\frac{3\beta c_0^2 + 8(\alpha+\rho^2)e^{2\rho t}}{2\sqrt{2}\sqrt{\alpha+\rho^2}\sqrt{5\gamma c_0^4 + 6\beta c_0^2 e^{2\rho t} + 8(\alpha+\rho^2)e^{4\rho t}}}\right) - 3\sqrt{10}\beta e^{2\rho t}\tanh^{-1}\left(\frac{5\gamma c_0^2 + 3\beta e^{2\rho t}}{\sqrt{5}\sqrt{\gamma}\sqrt{5\gamma c_0^4 + 6\beta c_0^2 e^{2\rho t} + 8(\alpha+\rho^2)e^{4\rho t}}}\right)\right). \tag{53}$$

The value of c_0 is found from the initial condition $x'(0) = \dot{x}_0$, and it is a solution to the sextic

$$-8\left(\alpha x_0^2 + 2\rho^2 x_0^2 + 2\rho\dot{x}_0 x_0 + \dot{x}_0^2\right) + 2\left(4\alpha + 4\rho^2 - 3\beta x_0^2\right)z^2 + \left(6\beta - 5\gamma x_0^2\right)z^4 + 5\gamma z^6 = 0. \tag{54}$$

The number ρ is a free parameter that is chosen in order to minimize the residual error. In particular, when $\varepsilon \longrightarrow 0$, we obtain approximate trigonometric solution to the undamped cubic-quintic Duffing equation

$$\ddot{x} + \alpha x + \beta x^3 + \gamma x^5 = 0, x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{55}$$

Example 2. Let us consider the i.v.p.

$$\ddot{x} + 0.25|\dot{x}|\dot{x} + x + 5x^3 + 10x^5 = 0, x(0) = 0 \text{ and } x'(0) = 0.1. \tag{56}$$

The approximate analytical solution for $\rho = 0.0084$ is

$$u_{\text{approx}}(t) = x(t) = -0.0982083e^{-0.0084t}\cos\left(f(t) + \frac{\pi}{2}\right), \tag{57}$$

where the function $f(t)$ is given by (52) with $\varepsilon = 0.25$, $\alpha = 1$, $\beta = 5$, $\gamma = 10$, $c_0 = -0.0983$, and $\rho = 0.0084$ (see Figure 2).

The obtained results may be applied to solve the pendulum equation with quadratic damping

$$\ddot{\theta} + \varepsilon\dot{\theta}|\dot{\theta}| + \omega^2 \sin\theta = 0, \theta(0) = \theta_0 \text{ and } \theta'(0) = \dot{\theta}_0. \tag{58}$$

Indeed, we may use the approximation

$$\sin\theta \approx \theta - \frac{\theta^3}{6} + \frac{\theta^5}{131} \text{ for} |x| \le \frac{\pi}{2}, \tag{59}$$

and then we replace i.v.p. (58) with the i.v.p.

$$\ddot{\theta} + \varepsilon\dot{\theta}|\dot{\theta}| + \omega^2\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{131}\right) = 0, \theta(0) = \theta_0 \text{ and } \theta'(0) = \dot{\theta}_0. \tag{60}$$

Example 3. Let us consider the i.v.p.

$$\ddot{\theta} + 0.2\dot{\theta}|\dot{\theta}| + \sin\theta = 0, \theta = (0) = 20^\circ \text{ and } \theta'(0) = 0.1. \tag{61}$$

The optimal value for ρ is $\rho = 0.0225$. The value of c_0 is $c_0 = -0.365609$ (see Figure3).

2.6.2. Second Approach. Let us consider the i.v.p.

$$\ddot{x} + \varepsilon\dot{x}|\dot{x}| + \alpha x + \beta x^3 + \gamma x^5 = 0, x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{62}$$

Assume the ansatz

$$x(t) = x_{\rho,\kappa}(t) = e^{-\rho t}y(\tau), \tau = \tau(t) = \frac{1 - \exp(-\varepsilon\kappa t)}{\varepsilon\kappa}, \tag{63}$$

where the function $y = y(\tau)$ is the exact solution to the i.v.p.

$$\ddot{y}(\tau) + \alpha y(\tau) + \beta y^3(\tau) + \gamma y^5(\tau) = 0, y(0) = x_0 \text{ and } x'(0) = \dot{x}_0 + \rho x_0. \tag{64}$$

The numbers ρ and κ are free parameters that are chosen in order to minimize the residual error

$$R(t) = \ddot{x}(t) + \varepsilon\dot{x}(t)|\dot{x}(t)| + \alpha x(t) + \beta x^3(t) + \gamma x^5(t). \tag{65}$$

Observe that when $\varepsilon \longrightarrow 0$, $((1 - \exp(-\varepsilon\kappa t))/\varepsilon\kappa) \longrightarrow t$ and then we obtain the exact solution to the undamped and

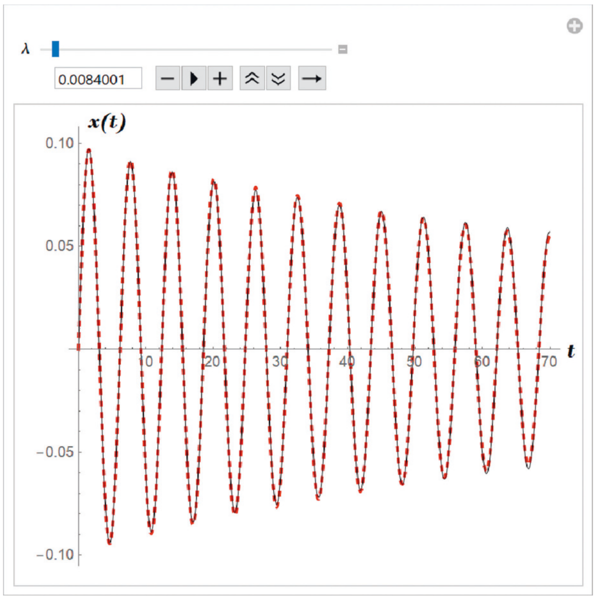

Figure 2: Comparison between analytical approximant (57) and the Runge–Kutta numerical solution for i.v.p. (56).

unforced cubic-quintic oscillator (3). So, we expect accurate approximate analytical solution for small ε. This approach is more accurate, but here the solution involves elliptic functions and the solution is not elementary.

2.6.3. Third Approach. Let us consider the i.v.p.

$$\ddot{x}+\varepsilon\dot{x}|\dot{x}|+\alpha x+\beta x^3+\gamma x^5=0,\ x(0)=x_0 \text{ and } x'(0)=\dot{x}_0. \tag{66}$$

Assume the ansatz

$$x(t)=x_{\rho,\lambda}(t)=ce^{-\rho t}\frac{\cos(f(t)+d)}{\sqrt{1+\lambda\cos^2(f(t)+d)}}. \tag{67}$$

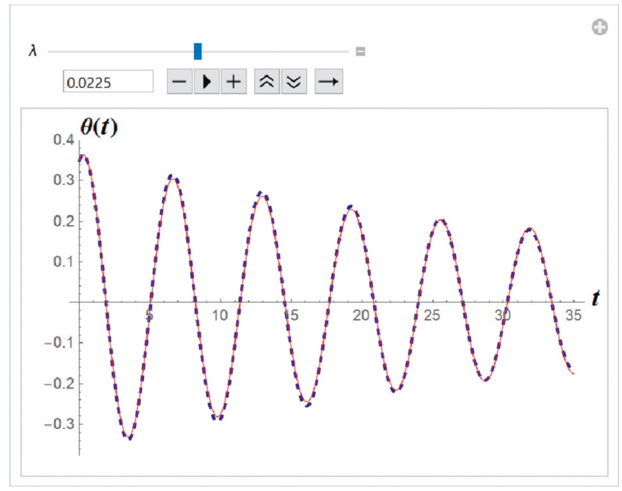

Figure 3: Comparison between the analytical approximant and the Runge–Kutta numerical solution for i.v.p. (61) with $\rho=0.0225$ and $c_0=-0.365609$.

Proceeding in a similar way as in the first approach, we may choose

$$f(t)=\int_0^t\sqrt{\frac{2(\lambda+2)(\alpha+\rho^2)+\beta c^2e^{-2\rho\tau}}{10\lambda+4}}d\tau. \tag{68}$$

Here we have two free parameters ρ and λ that are chosen in order to get as less residual error as possible. The numbers c and d are determined from the initial conditions as follows:

$$d=\pm\cos^{-1}\left(\pm\frac{x_0}{\sqrt{c^2-\lambda x_0^2}}\right). \tag{69}$$

The number c is a solution to the octic

$$\begin{aligned}&-2\lambda^2(\lambda+1)(\lambda+2)x_0^6(\alpha+\rho^2)+\lambda x_0^4\left(6\alpha\lambda^2+16\alpha\lambda+8\alpha+6\lambda^2\rho^2+16\lambda\rho^2+8\rho^2-\beta\lambda^2x_0^2-\beta\lambda x_0^2\right)c^2\\&+\begin{pmatrix}-6\alpha\lambda^2x_0^2-14\alpha\lambda x_0^2-4\alpha x_0^2+3\beta\lambda^2x_0^4+2\beta\lambda x_0^4-\\6\lambda^2\rho^2x_0^2-24\lambda\rho^2x_0^2-20\lambda\rho\dot{x}_0x_0-10\lambda\dot{x}_0^2-8\rho^2x_0^2-8\rho\dot{x}_0x_0-4\dot{x}_0^2\end{pmatrix}c^4\\&+\left(2\alpha\lambda+4\alpha+2\lambda\rho^2+4\rho^2-3\beta\lambda x_0^2-\beta x_0^2\right)c^6+\beta c^8=0.\end{aligned} \tag{70}$$

2.7. Third Case: Quadratically Damped and Forced Oscillator. Let us consider the i.v.p.

$$\ddot{x}+\varepsilon\dot{x}|\dot{x}|+\alpha x+\beta x^3+\gamma x^5=F_0\cos(\omega t)+F_1\sin(\omega t),\quad x(0)=x_0 \text{ and } x'(0)=\dot{x}_0. \tag{71}$$

Assume the ansatz

$$x(t)=u(t)+d_0\cos(\omega t)+d_1\sin(\omega t), \tag{72}$$

where the function $u=u(t)$ is a solution to some i.v.p.

$$\ddot{u}+\varepsilon\dot{u}|\dot{u}|+pu+qu^3+\gamma u^5=0,\ u(0)=x_0-d_0 \text{ and } u'(0)=\dot{x}_0-d_1\omega. \tag{73}$$

The suitable constants p, q, r, d_0, and d_1 are to be determined. Define the residual function

$$\begin{aligned}R(t)=&\ddot{x}(t)+\varepsilon\dot{x}(t)|\dot{x}(t)|+\alpha x(t)+\beta x^3(t)+\gamma x^5(t)\\&-F_0\cos(\omega t)-F_1\sin(\omega t).\end{aligned} \tag{74}$$

The expression $R(t)$ contains many terms. Equating some coefficients of $v(t)$, $v'(t)$, $\sin(j\omega t)$, and $\cos(j\omega t)$ ($j = 1, 2, 3, \ldots$), we obtain the following algebraic system:

$$8\alpha d_1 + 6\beta d_1^3 + 6\beta d_0^2 d_1 + 5\gamma d_1^5 + 10\gamma d_0^2 d_1^3 + 5\gamma d_0^4 d_1 - 8d_1\omega^2 - 8F_1 = 0, \tag{75}$$

$$8\alpha d_0 + 6\beta d_0^3 + 6\beta d_1^2 d_0 + 5\gamma d_0^5 + 10\gamma d_1^2 d_0^3 + 5\gamma d_1^4 d_0 - 8d_0\omega^2 - 8F_0 = 0, \tag{76}$$

$$8\alpha + 12\beta d_0^2 + 12\beta d_1^2 + 15\gamma d_0^4 + 30\gamma d_1^2 d_0^2 + 15\gamma d_1^4 - 8p = 0, \tag{77}$$

$$\beta + 5\gamma d_0^2 + 5\gamma d_1^2 - q = 0, \tag{78}$$

$$\gamma - r = 0. \tag{79}$$

From equations (77)–(79), we obtain

$$.\,p = \alpha + \frac{3}{8}\left(d_0^2 + d_1^2\right)\left(4\beta + 5\gamma\left(d_0^2 + d_1^2\right)\right), \quad q = \beta + 5\gamma\left(d_0^2 + d_1^2\right), \quad r = \gamma. \tag{80}$$

Decupling equations (75) and (76) by means of their eliminants gives

$$\left(5\gamma F_0^4 + 10\gamma F_1^2 F_0^2 + 5\gamma F_1^4\right)d_0^5 + \left(6\beta F_0^4 + 6\beta F_1^2 F_0^2\right)d_0^3 + \left(8\alpha F_0^4 - 8F_0^4\omega^2\right)d_0 - 8F_0^5 = 0, \tag{81}$$

$$\left(5\gamma F_0^4 + 10\gamma F_1^2 F_0^2 + 5\gamma F_1^4\right)d_1^5 + \left(6\beta F_1^4 + 6\beta F_0^2 F_1^2\right)d_1^3 + \left(8\alpha F_1^4 - 8F_1^4\omega^2\right)d_1 - 8F_1^5 = 0. \tag{82}$$

We choose the least in magnitude real root d_0 to quintic (81) and the least in magnitude real root d_1 to quintic (82). Assuming that the forces F_0 and F_1 are small in magnitude, we have the following approximations for these roots:

$$d_0 = \frac{4F_0\left(\alpha - \omega^2\right)^2}{4\left(\alpha - \omega^2\right)^3 + 3\beta\left(F_0^2 + F_1^2\right)}, \quad d_1 = \frac{4F_1\left(\alpha - \omega^2\right)^2}{4\left(\alpha - \omega^2\right)^3 + 3\beta\left(F_0^2 + F_1^2\right)}. \tag{83}$$

More precise approximations are

$$d_0 = \frac{2F_0\sqrt{\alpha - \omega^2}}{\left(\alpha - \omega^2\right)^{3/2} \pm \sqrt{\left(\alpha - \omega^2\right)^3 + 3\beta\left(F_0^2 + F_1^2\right)}}, \quad d_1 = \frac{2F_1\sqrt{\alpha - \omega^2}}{\left(\alpha - \omega^2\right)^{3/2} \pm \sqrt{\left(\alpha - \omega^2\right)^3 + 3\beta\left(F_0^2 + F_1^2\right)}}. \tag{84}$$

3. Further Applications

Suppose we are given a quadratically damped oscillator

$$\ddot{x} + \varepsilon\dot{x}|\dot{x}| + h(x) = 0, x(0) = x_0 \text{ and } x'(0) = \dot{x}_0, \tag{85}$$

where $h(-x) = -h(x)$ is an odd continuous function. Suppose that $|x| \le M$. Then, we may approximate the function $h(x)$ by means of the following quintic polynomial:

$$h(x) \approx \alpha x + \beta x^3 + \gamma x^5, \tag{86}$$

where

$$\alpha = \frac{h(-(M/\sqrt{2})) - h(M/\sqrt{2}) - (5+3\sqrt{3})h(-(1/2)\sqrt{2-\sqrt{3}}\,M) + (5+3\sqrt{3})h((1/2)\sqrt{2-\sqrt{3}}\,M) - (3\sqrt{3}-5)(h(-(1/2)\sqrt{2+\sqrt{3}}\,M) - h((1/2)\sqrt{2+\sqrt{3}}\,M))}{3\sqrt{2}M}, \tag{87}$$

$$\beta = \frac{\sqrt{2}(-8h(-(M/\sqrt{2})) + 8h((M/\sqrt{2})) + (7+5\sqrt{3})h(-(1/2)\sqrt{2-\sqrt{3}}\,M) - (7+5\sqrt{3})h((1/2)\sqrt{2-\sqrt{3}}\,M) + (5\sqrt{3}-7)(h(-(1/2)\sqrt{2+\sqrt{3}}\,M) - h((1/2)\sqrt{2+\sqrt{3}}\,M)))}{3M^3}, \tag{88}$$

$$\gamma = \frac{4\sqrt{2}(2h(-(M/\sqrt{2})) - 2h((M/\sqrt{2})) - (1+\sqrt{3})h(-(1/2)\sqrt{2-\sqrt{3}}\,M) + (1+\sqrt{3})h((1/2)\sqrt{2-\sqrt{3}}\,M) - (\sqrt{3}-1)(h(-(1/2)\sqrt{2+\sqrt{3}}\,M) - h((1/2)\sqrt{2+\sqrt{3}}\,M)))}{3M^5}. \tag{89}$$

Then, the i.v.p. (85) is reduced to the i.v.p. (1).

Example 4. Consider the motion of a satellite along a path that is equidistant from two identical massive stars with mutually interacting gravitational fields. If the distance between the two stars is 2d and the coordinate of the satellite motion is x, then the equation of motion of the satellite is given as

$$\ddot{x} + \varepsilon|\dot{x}|\dot{x} + \frac{2mx}{\left(d^2 + x^2\right)^{3/2}} = 0, x(0) = x_0 \text{ and } x'(0) = \dot{x}_0, \tag{90}$$

where m is the mass of a star and the restoring force is

$$h(x) = \frac{2mx}{\left(d^2 + x^2\right)^{3/2}}. \tag{91}$$

The nonlinear restoring force is an irrational force because of the bottom square root. The restoring force spikes near the origin. The spikes indicate the point when the satellite is most influenced by the mutual gravitational field of the stars. Away from the origin, the restoring force decreases gradually and approaches the horizontal axis asymptotically. This means that the satellite is far away from the stars and experiences a much smaller gravitational force.

4. Conclusions

We have obtained approximate analytical solutions to the quadratically damped Duffing oscillator equation by means of an elementary approach. We introduced a parameter technique that allowed us to optimize the obtained solution. The results are also valid for the linear quadratically damped oscillator $\ddot{x} + \varepsilon\dot{x}|\dot{x}| + \alpha x = 0$. Also, a more general quadratically damped oscillator $\ddot{x} + \varepsilon\dot{x}|\dot{x}| + h(x) = 0$ may be solved for any odd parity function $h(x)$. We also show the way to solve quadratically damped forced oscillators having the form $\ddot{x} + \varepsilon\dot{x}|\dot{x}| + h(x) = F(t)$ for any continuous functions $h(x)$ and $F(t)$ with $h(-x) = -h(x)$.

The quadratically damped cubic-quintic oscillator having both forcing term and quadratic damping term has been analyzed analytically using some highly accurate approaches. The proposed analytical techniques may be applied to solve other strongly nonlinear oscillators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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