Research Article
The Galerkin Method for Solving Strongly Nonlinear Oscillators

Alvaro H. S. Salas 🆔

Universidad Nacional de Colombia, Fizmako Research Group, Bogotá, Colombia

Correspondence should be addressed to Alvaro H. S. Salas; ahsalass@unal.edu.co

In this paper, we make use of the Galerkin method for solving nonlinear second-order ODEs that are related to some strongly nonlinear oscillators arising in physics and engineering. We derive the iterative schemes for finding the coefficients that appear in the linear Galerkin hat combination in the ansatz form solution. These coefficients may be found iteratively by solving either a quadratic or a higher degree algebraic equation. Examples are presented to illustrate the obtained results. Some exact solutions are given, and they are compared with both the Runge–Kutta numerical solution and the solution obtained using the Galerkin finite element method.

1. Introduction

The nonlinear equation describing an oscillator with a cubic nonlinearity is called the Duffing equation. This equation has a variety of applications in science and engineering, early mechanical failure signal, nonlinear circuit design [1], image processing [2], vibration of buckled beams [3], solitons [4–6], chaos [7], and many areas of physics.

There are many methods for solving nonlinear differential equations. In this paper, we concentrate on the numerical solution to the Duffing equation by means of the finite element method. This method is due to Galerkin, a Russian engineer and scientist. We also derive formulas for solving a wide class of nonlinear oscillators. The Galerkin solutions are compared with the solutions obtained using the Runge–Kutta numerical method.

2. Finite Element Method or Galerkin Hat Method

Let us consider a polynomial second-order damped and forced ode:

$$\begin{cases} \ddot{x} + 2\varepsilon\dot{x} + P(x) = f(t), \\ x(0) = 0, x'(0) = \dot{x}_0, \end{cases} \tag{1}$$

where $x \equiv x(t)$ and $P \equiv P(x)$ is a polynomial whose coefficients depend on t, say

$$P(x) = \sum_{s=0}^{M} a_s(t)x^s, x = x(t). \tag{2}$$

Given the i.v.p.

$$\begin{cases} \ddot{y} + 2\varepsilon\dot{y} + Q(y) = f(t), \\ y(0) = y_0 \neq 0, \quad \text{and} \quad y'(0) = \dot{y}_0, \\ \qquad \text{where} \\ Q(y) = \sum_{r=0}^{M} b_r(t)y^r, y = y(t). \end{cases} \tag{3}$$

we define

$$y(t) = y_0 + x(t),$$
$$x(0) = 0,$$
$$x'(0) = y_0.$$
(4)

Then,

$$Q(y) = \sum_{r=0}^{M} b_r (y_0 + x(t))^r$$

$$= \sum_{r=0}^{M} b_r \sum_{s=0}^{r} \binom{r}{s} y_0^{r-s} x^s(t) = \sum_{s=0}^{M} \left(\sum_{r=s}^{M} \binom{r}{s} y_0^{r-s} b_r \right) x^s(t),$$

$$= \sum_{s=0}^{M} a_s x^s =: P(x), x = x(t),$$
(5)

where

$$a_s = \sum_{r=s}^{M} \binom{r}{s} y_0^{r-s} b_r.$$

Thus, the problem reduces to (1). Some particular cases to i.v.p. (1) are

$$\begin{cases} \ddot{x} + 2\varepsilon \dot{x} + n(t) + p(t)x = f(t), \\ \ddot{x} + 2\varepsilon \dot{x} + n(t) + p(t)x + q(t)x^2 = f(t), \\ \ddot{x} + 2\varepsilon \dot{x} + n(t) + p(t)x + q(t)x^2 + r(t)x^3 = f(t). \end{cases}$$
(6)

We will use the same idea as for the linear case

$$\begin{cases} \ddot{x} + 2\varepsilon \dot{x} + p(t)x = f(t), \\ x(0) = 0, x'(0) = \dot{x}_0, \end{cases}$$
(7)

that is, we will assume an approximate analytical solution in the ansatz form

$$x = x(t) = \sum_{k=1}^{n} c_k \varphi_k(t),$$
(8)

where the functions $\varphi_k(t)$ are the so-called linear Galerkin hats. Let $0 \le t \le T$. Choose some positive integer $n \ge 2$ and define the step $h = T/n$ and let $\xi_j = jh = jT/n$ for $j = 0, 1, 2, \ldots$ The functions $\varphi_k(t)$ are defined as follows.

$$\varphi_k(t) = \begin{cases} \dfrac{t - \xi_{k-1}}{h} & if \quad \xi_{k-1} \le t \le \xi_k, \\ & \quad\quad\quad\quad\quad\quad\quad for \, k = 1, 2, \ldots, n. \\ -\dfrac{t - \xi_{k+1}}{h} & if \quad \xi_k \le t \le \xi_{k+1}, \end{cases}$$
(9)

For an illustration, see Figure 1.

Some properties of these functions can be illustrated as follows.

$$\varphi_j(t)\varphi_k(t)dt = 0,$$
(10)

for $|j - k| \ge 2$ and $t \in [0, T]$, and

$$\int_0^T \varphi_j^p(t) = \frac{2T}{(p+1)n},$$
(11)

for $j \ge 1$ and $p = 1, 2, 3, \ldots$.

In general,

$$\int_0^T \varphi_j^r(t)\varphi_k^s(t) = \frac{Tr!s!}{n(r+s+1)!}, \quad for|j-k| = 1,$$

$$r, s = 0, 1, 2, 3, \ldots.$$
(12)

Using the formula

$$\int_0^T \varphi_j(t)x(t)^N dt = \frac{h}{(N+1)(N+2)} \left(\sum_{k=0}^{N-1} (k+1)\left(c_{j-1}^{N-k} + c_{j+1}^{N-k}\right)c_j^k + 2(N+1)c_j^N \right) \text{for any } N \ge 0,$$
(13)

$$c_0 = c_{n+1} = 0,$$

and assuming that $a_j(t) \equiv a_j = $ const, we may evaluate easily the following integral:

$$\int_0^T P(x)\varphi_j(t)dt = \sum_{N=0}^{m} a_N \int_0^T \varphi_j(t)x(t)^N dt \text{ for any } j.$$
(14)

Moreover,

$$\int_0^T x\prime(t)\varphi_j(t)dt = -\int_0^T x\prime(t)\varphi_j'(t)dt = \frac{c_{j-1} - 2c_j + c_{j+1}}{h} \text{ for any } j,$$

$$\int_0^T x\prime(t)\varphi_j(t)dt = \frac{1}{2}\left(c_{j+1} - c_{j-1}\right).$$
(15)

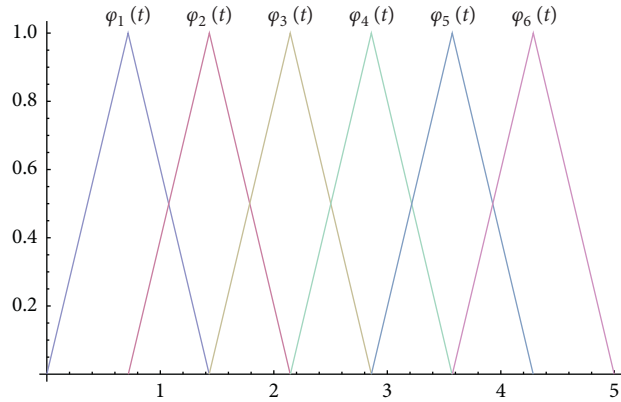Thus, for example, if $a_0, a_1, \ldots$ are independent of t,

FIGURE 1: Galerkin hats for $n = 7$ and $T = 5$.

$$\int_0^T \left[x''(t) + \delta x'(t) + a_0 + a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) + a_4 x^4(t) + a_5 x^5(t) \right] \varphi_j(t) dt = \frac{n\left(c_{j-1} - 2c_j + c_{j+1}\right)}{T}$$

$$+ \frac{\delta}{2}\left(c_{j+1} - c_{j-1}\right) + \frac{T}{n} a_0 + a_1 A_1 + a_2 A_2 + a_3 A_3 + a_4 A_4 + a_5 A_5), \tag{16}$$

where

$$A_1 = \frac{T}{6n}\left(c_{j-1} + 4c_j + c_{j+1}\right),$$

$$A_2 = \frac{T}{12n}\left(c_{j-1}^2 + 2c_j c_{j-1} + 6c_j^2 + c_{j+1}^2 + 2c_j c_{j+1}\right),$$

$$A_3 = \frac{T}{20n}\left(c_{j-1}^3 + 2c_j c_{j-1}^2 + 3c_j^2 c_{j-1} + 8c_j^3 + c_{j+1}^3 + 2c_j c_{j+1}^2 + 3c_j^2 c_{j+1}\right), \tag{17}$$

$$A_4 = \frac{T}{30n}\left(c_{j-1}^4 + 2c_j c_{j-1}^3 + 3c_j^2 c_{j-1}^2 + 4c_j^3 c_{j-1} + 10c_j^4 + c_{j+1}^4 + 2c_j c_{j+1}^3 + 3c_j^2 c_{j+1}^2 + 4c_j^3 c_{j+1}\right),$$

$$A_5 = \frac{T}{42n}\left(c_{j-1}^5 + 2c_j c_{j-1}^4 + 3c_j^2 c_{j-1}^3 + 4c_j^3 c_{j-1}^2 + 5c_j^4 c_{j-1} + 12c_j^5 + c_{j+1}^5 + 2c_j c_{j+1}^4 + 3c_j^2 c_{j+1}^3 + 4c_j^3 c_{j+1}^2 + 5c_j^4 c_{j+1}\right).$$

In general, if a_r does not depend on t,

$$\int_0^T \left(x''(t) + \delta x'(t) + \sum_{s=0}^M a_s x^s(t) \right) \varphi_j(t) dt = \frac{n\left(c_{j-1} - 2c_j + c_{j+1}\right)}{T} + \frac{1}{2}\delta\left(c_{j+1} - c_{j-1}\right) + \frac{Ta_0}{n} + \sum_{s=0}^M A_s a_s, \tag{18}$$

where

$$A_s = \frac{T}{n(s+1)(s+2)} \sum_{k=0}^{s-1} \left((k+1)\left(c_{j-1}^{s-k} + c_{j+1}^{s-k}\right)c_j^k + 2(s+1)c_j^s\right). \tag{19}$$

Let us consider the forced and damped oscillator (1). Following are useful expressions for different forces:

$$\int_0^T f(t)\varphi_j(t)dt = \frac{4n}{T\Omega^2}\sin^2\left(\frac{T\Omega}{2n}\right)\left(\frac{F_0 T}{n} + F_1\cos\left(\frac{jT\Omega}{n}\right) + F_2\sin\left(\frac{jT\Omega}{n}\right)\right),$$

$$f(t) = f_0 + f_1\cos(\Omega t) + f_2\sin(\Omega t),$$

$$\Omega \neq 0,$$

$$\int_0^T f(t)\varphi_j(t)dt = \frac{ne^{-(j+1)\lambda T/n}}{T(\lambda^2+\Omega^2)^2}\left(\begin{array}{c} -2\lambda\Omega\sin\left(\frac{(j+1)T\Omega}{n}\right) + (\lambda-\Omega)(\lambda+\Omega)e^{\lambda T/n}\cos\left(\frac{(j-1)T\Omega}{n}\right)+ \\[4pt] (\lambda-\Omega)(\lambda+\Omega)\cos\left(\frac{(j+1)T\Omega}{n}\right)- \\[4pt] 2e^{\lambda T/n}\left(\lambda\Omega\left(e^{\lambda T/n}\sin\left(\frac{(j-1)T\Omega}{n}\right) - 2\sin\left(\frac{jT\Omega}{n}\right)\right) + (\lambda-\Omega)(\lambda+\Omega)\cos\left(\frac{jT\Omega}{n}\right)\right) \end{array}\right),$$

$$f(t) = e^{-\lambda t}\cos(\Omega t),$$

$$|\Omega|^2 + |\lambda|^2 \neq 0,$$

$$\int_0^T f(t)\varphi_j(t)dt = \frac{ne^{-(j+1)\lambda T/n}}{T(\lambda^2+\Omega^2)^2}\left(\begin{array}{c} (\lambda-\Omega)(\lambda+\Omega)e^{\lambda T/n}\left(e^{\lambda T/n}\sin\left(\frac{(j-1)T\Omega}{n}\right) - 2\sin\left(\frac{jT\Omega}{n}\right)\right)+ \\[4pt] (\lambda-\Omega)(\lambda+\Omega)\sin\left(\frac{(j+1)T\Omega}{n}\right) + 2\lambda\Omega e^{2\lambda T/n}\cos\left(\frac{(j-1)T\Omega}{n}\right)- \\[4pt] 4\lambda\Omega e^{\lambda T/n}\cos\left(\frac{jT\Omega}{n}\right) + 2\lambda\Omega\cos\left(\frac{(j+1)T\Omega}{n}\right) \end{array}\right),$$

$$f(t) = e^{-\lambda t}\sin(\Omega t),$$

$$|\Omega|^2 + |\lambda|^2 \neq 0,$$

$$\int_0^T f(t)\varphi_j(t)dt = \frac{\lambda^{-\rho-2}}{T}\left(\begin{array}{c} \lambda T\left((1-j)\Gamma\left(\rho+1,\frac{(j-1)T\lambda}{n}\right) + 2j\Gamma\left(\rho+1,\frac{jT\lambda}{n}\right) - (j+1)\Gamma\left(\rho+1,\frac{(j+1)T\lambda}{n}\right)\right)+ \\[4pt] n\left(\Gamma\left(\rho+2,\frac{(j-1)T\lambda}{n}\right) - 2\Gamma\left(\rho+2,\frac{jT\lambda}{n}\right) + \Gamma\left(\rho+2,\frac{(j+1)T\lambda}{n}\right)\right) \end{array}\right),$$

$$f(t) = e^{-\lambda t}t^\rho,$$

$$\int_0^T f(t)\varphi_j(t)dt \approx \frac{n}{9Tw}\left((A-1)\left(\cos\left(\frac{3(j-1)T\sqrt{w}}{n}\right) - 2\cos\left(\frac{3jT\sqrt{w}}{n}\right) + \cos\left(\frac{3(j+1)T\sqrt{w}}{n}\right)\right) - 18A\left(\cos\left(\frac{T\sqrt{w}}{n}\right) - 1\right)\cos\left(\frac{jT\sqrt{w}}{n}\right)\right),$$

$$f(t) = cn(\Omega, m) \approx A\cos(\sqrt{w}t) + (1-A)\cos(3\sqrt{w}t).w = \frac{(27m^2 - 92m + 64)\Omega^2}{3m^2 - 60m + 64}.A = \frac{2(15m^2 - 48m + 32)}{27m^2 - 92m + 64}, |m| \leq 0.5,$$

$$\int_0^T f(t)\varphi_j(t)dt \approx \frac{n}{9Tw}\left((A-1)\left(\cos\left(\frac{3(j-1)T\sqrt{w}}{n}\right) - 2\cos\left(\frac{3jT\sqrt{w}}{n}\right) + \cos\left(\frac{3(j+1)T\sqrt{w}}{n}\right)\right) - 18A\left(\cos\left(\frac{T\sqrt{w}}{n}\right) - 1\right)\cos\left(\frac{jT\sqrt{w}}{n}\right)\right),$$

$$f(t) = cn(\Omega t, m) \approx A\cos(\sqrt{w}t) + (1-A)\cos(3\sqrt{w}t).w$$

$$= \frac{(27m^2 - 92m + 64)\Omega^2}{3m^2 - 60m + 64}.A = \frac{2(15m^2 - 48m + 32)}{27m^2 - 92m + 64}, \quad |m| \leq 0.5,$$

$$\int_0^T f(t)\varphi_j(t)dt$$

$$\approx \frac{(3-4A)^2 n(18A(\cos(T\Omega/3n - 4An) - 1)\sin(jT\Omega/3n - 4An) - (A-1)(\sin(3(j-1)T\Omega/(4A-3)n) + \sin(3(j+1)T\Omega/(4A-3)n) + 2\sin(3jT\Omega/3n - 4An)))}{9T\Omega^2},$$

$$f(t) = sn(\Omega t, m) \approx A\sin\left(\frac{t\Omega}{4A-3}\right) - (1-A)\sin\left(\frac{3t\Omega}{4A-3}\right),$$

$$A = -\frac{-25m^2 + 144m - 128}{29m^2 - 152m + 128}, \quad |m| \leq 0.25.$$

(20)

Other useful formulas when the a_j depend on time are

$$\int_0^T tx(t)\varphi_j(t)dt = \frac{1}{12}h\big(c_{j-1}(2a + h(2j-1)) + 8c_j(a + hj) + c_{j+1}(2a + 2hj + h)\big),$$

$$\int_0^T t^2 x(t)\varphi_j(t)dt = \frac{1}{60}h^3\big((40j^2 + 4)c_j + (10(j-1)j + 3)c_{j-1} + (10j(j+1) + 3)c_{j+1}\big),$$

$$\int_0^T t^3 x(t)\varphi_j(t)dt = \frac{1}{60}h^4\big(4j(10j^2 + 3)c_j + (2j-1)(5(j-1)j + 2)c_{j-1} + (2j+1)(5j(j+1) + 2)c_{j+1}\big),$$

$$\int_0^T tx(t)^2\varphi_j(t)dt = \frac{1}{60}h^2\big((5j-3)c_{j-1}^2 + 2(5j-2)c_j c_{j-1} + 30jc_j^2 + (5j+3)c_{j+1}^2 + 2(5j+2)c_j c_{j+1}\big),$$

$$\int_0^T t^2 x(t)^2\varphi_j(t)dt = \frac{1}{60}h^3\left(\begin{array}{c}(30j^2 + 2)c_j^2 + (j(5j-6) + 2)c_{j-1}^2 + \\[6pt] 2(j(5j-4) + 1)c_j c_{j-1} + (j(5j+6) + 2)c_{j+1}^2 + 2(j(5j+4) + 1)c_j c_{j+1}\end{array}\right),$$

$$\int_0^T t^3 x(t)^2\varphi_j(t)dt = \frac{1}{420}h^4\left(\begin{array}{c}42(5j^3 + j)c_j^2 + 7j(j(5j-9) + 6-)10)c_{j-1}^2 + 2(7j(j(5j-6) + 3) - 4)c_j c_{j-1} + \\[6pt] (7j(j(5j+9) + 6) + 10)c_{j+1}^2 + 2(7j(j(5j+6) + 3) + 4)c_j c_{j+1}\end{array}\right), \tag{21}$$

$$\int_0^T tx(t)^3\varphi_j(t)dt = \frac{1}{60}h^2\left(\begin{array}{c}(3j-2)c_{j-1}^3 + 3(2j-1)c_j c_{j-1}^2 + 3(3j-1)c_j^2 c_{j-1} + \\[6pt] 24jc_j^3 + (3j+2)c_{j+1}^3 + 3(2j+1)c_j c_{j+1}^2 + 3(3j+1)_j^2)c_{j+1}\end{array}\right),$$

$$\int_0^T t^2 x(t)^3\varphi_j(t)dt = \frac{1}{420}h^3\left(\begin{array}{c}8(21j^2 + 1)c_j^3 + (7j(3j-4) + 10)c_{j-1}^3 + 6(7(j-1)j + 2)c_j c_{j-1}^2 + \\[6pt] 3(7j(3j-2) + 3)c_j^2 c_{j-1} + (7j(3j+4) + 10)c_{j+1}^3 + \\[6pt] 6(7j(j+1) + 2)c_j c_{j+1}^2 + 3(7j(3j+2) + 3)c_j^2 c_{j+1}\end{array}\right),$$

$$\int_0^T t^3 x(t)^3\varphi_j(t)dt = \frac{1}{280}h^4\left(\begin{array}{c}16(7j^3 + j)c_j^3 + (2j(7(j-2)j + 10) - 5)_{j-1}^3 + (2j-1)(14(j-1)j + 5)c_j c_{j-1}^2 + \\[6pt] 3(2j(7(j-1)j + 3) - 1)c_j^2 c_{j-1} + (2j(7j(j+2) + 10) + 5)c_{j+1}^3 + \\[6pt] (2j+1)(14j(j+1) + 5)c_j c_{j+1}^2 + 3(2j(7j(j+1) + 3) + 1)c_j^2 c_{j+1}\end{array}\right).$$

Other formulas for calculating are given in the Appendix.

3. Applications

3.1. Linear Oscillator. This is the ode

$$\begin{aligned}\ddot{x} + \alpha + \beta x &= 0, \\ \beta &\neq 0, \\ x(0) &= 0, \\ x'(0) &= \dot{x}_0, \\ 0 &\leq t \leq T.\end{aligned} \tag{22}$$

The exact solution is given by

$$x(t) = \frac{1}{\beta}\big(\alpha(\cos(\sqrt{\beta}\,t) - 1) + \sqrt{\beta}\,\dot{x}_0 \sin(\sqrt{\beta}\,t)\big). \tag{23}$$

Assume the ansatz

$$x = x(t) = \sum_{k=1}^n c_k \varphi_k(t). \tag{24}$$

Define $c_{-1} = c_0 = 0, c_1 = (T/n)\dot{x}_0$ and

$$R_j = \int_0^T (\ddot{x} + \alpha + \beta x)\varphi_j(t)\mathrm{d}t, \quad j = 0, 1, 2, \ldots, n-1. \quad (25)$$

Then,

$$R_j = \frac{6\alpha T^2}{6nT} + \frac{6n^2 + \beta T^2}{6nT}c_{j-1} + \frac{4c(\beta T^2 - 3n^2)}{6nT}c_j$$

$$+ \left(\frac{\beta T}{6n} + \frac{n}{T}\right)c_{j+1} = 0 \text{ for any } j. \quad (26)$$

This is a linear recurrence that may be solved in closed form:

$$c_j = \frac{1}{6\beta}\left(\begin{array}{c} -6\alpha + 3\alpha\left(\dfrac{T\left(\sqrt{3}\sqrt{\beta(\beta T^2 - 12n^2)} - 2\beta T\right) + 6n^2}{6n^2 + \beta T^2}\right)^j + \\[2em] 3\alpha\left(\dfrac{6n^2 - T\left(\sqrt{3}\sqrt{\beta(\beta T^2 - 12n^2)} + 2\beta T\right)}{6n^2 + \beta T^2}\right)^j + \\[2em] \dfrac{1}{n\sqrt{T^2/3 - 4n^2/\beta}}\left(\begin{array}{c}\left(\left(T\left(\sqrt{3}\sqrt{\beta(\beta T^2 - 12n^2)} - 2\beta T\right) + 6n^2\right)^j - \right. \\[1.5em] \left(6n^2 - T\left(\sqrt{3}\sqrt{\beta(\beta T^2 - 12n^2)} + 2\beta T\right)\right)^j \\[1.5em] \left.\left(6n^2 + \beta T^2\right)^{-j}\left(\dot{x}_0\left(6n^2 + \beta T^2\right) + 3\alpha nT\right)\right)\end{array}\right)\end{array}\right). \quad (27)$$

$$j = 1, 2, 3, \ldots, n$$

Example 1. Let $\alpha = 0$, $\beta = 1$, $\dot{x}_0 = 1$, and $T = 2\pi$. The exact solution to (14) is $x(t) = \sin(t)$. From (19), we obtain

$$c_j = \frac{\left((17440418\pi^2/n^2) + 26160627\right)}{2953\sqrt{78481881 - \left(26160627\pi^2/n^2\right)}}\sin\left(j\tan^{-1}\left(\frac{5906\pi\sqrt{78481881 - \left(26160627\pi^2/n^2\right)}}{n\left(26160627 - \left(34880836\pi^2/n^2\right)\right)}\right)\right). \quad (28)$$

In Figure 2, we compare the exact solution with the approximate solution (pairs $(2\pi/nj, c_j)$, $j = 1, 2, 3, \ldots, n$) for $n = 30$.

3.2. Undamped and Unforced Helmholtz Oscillator. This is the ode

The exact solution to (22) may be expressed in any of the following equivalent forms:
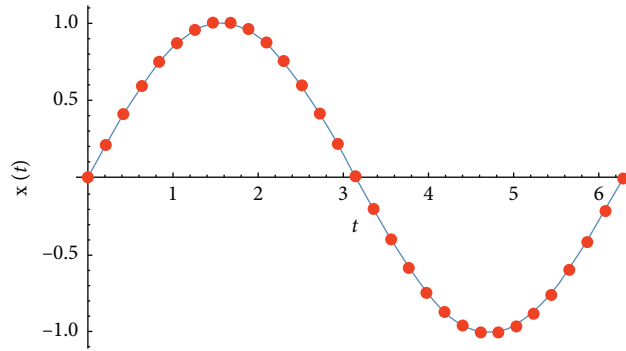
$$\ddot{y} + \alpha + \beta y + \gamma y^2 = 0,$$
$$y(0) = y_0,$$
$$y'(0) = \dot{y}_0,$$
$$0 \le t \le T. \quad (29)$$

Figure 2: Comparison between the Galerkin solution and the Runge–Kutta numerical solution.

$$y(t) = -\frac{\beta}{2\gamma} - \frac{6\wp\left(t + t_0; (1/12)(\beta^2 - 4\alpha\gamma), (1/216)\left((\beta + 2\gamma y_0)(\beta^2 - 6\alpha\gamma - 2\gamma y_0(\beta + \gamma y_0)) - 6\gamma^2 \dot{y}_0^2\right)\right)}{\gamma},$$

where

$$t_0 = \pm\wp^{-1}\left((1/12)(-\beta - 2\gamma y_0); (1/12)(\beta^2 - 4\alpha\gamma), (1/216)\left((\beta + 2\gamma y_0)(-6\alpha\gamma + \beta^2 - 2\gamma y_0(\beta + \gamma y_0)) - 6\gamma^2\dot{y}_0^2\right)\right),$$

$$y(t) = A - \frac{6(\alpha + A(A\gamma + \beta))}{12\wp\left(t + t_0; 1/12(\beta^2 - 4\alpha\gamma), 1/216(\beta + 2A\gamma)(\beta^2 - 2A\gamma\beta - 2\gamma(\gamma A^2 + 3\alpha))\right) + 2A\gamma + \beta},$$

(30)

where

$$-2\gamma A^3 + 3\beta A^2 + 6\alpha A + 6\alpha y_0 - 3\beta y_0^2 - 2\gamma y_0^3 - 3\dot{y}_0^2 = 0,$$

$$t_0 = \wp^{-1}\left(\frac{6\alpha + 4A^2\gamma + 5A\beta + 2A\gamma y_0 + \beta y_0}{12(A - y_0)}; \frac{1}{12}(\beta^2 - 4\alpha\gamma), \frac{1}{216}(2A\gamma + \beta)\left(-2\gamma(3\alpha + A^2\gamma) - 2A\beta\gamma + \beta^2\right)\right),$$

$$y(t) = A + \left(\frac{\sqrt{y_0 - A}\,cn(\sqrt{\omega}\,t, m) + b_1 sn(\sqrt{\omega}\,t, m)dn(\sqrt{\omega}\,t, m)}{1 + b_2 sn^2(\sqrt{\omega}\,t, m)}\right)^2,$$

where

$$b_1 = \frac{\dot{y}_0}{2\sqrt{\omega}\sqrt{y_0 - A}},$$

$$b_2 = \frac{\dot{y}_0^2 - 2(A - y_0)(\alpha + 2A\omega + y_0(\beta + \gamma y_0 - 2\omega))}{8\omega(A - y_0)^2},$$

$$\omega = -\frac{-2A^2\beta - 4A^2\gamma y_0 - 6\alpha A - 2A\beta y_0 + 2A\gamma y_0^2 + 6\alpha y_0 + 4\beta y_0^2 + 2\gamma y_0^3 + 3\dot{y}_0^2}{4(2m - 1)(A - y_0)^2},$$

(31)

$$m = \frac{1}{2}\left(1 - \frac{\sqrt{3}(2A\gamma + \beta)^2}{\sqrt{(2A\gamma + \beta)^2((\beta - 2A\gamma)(2A\gamma + 3\beta) - 16\alpha\gamma)}}\right),$$

$$2\gamma A^3 + 3\beta A^2 + 6\alpha A - 6\alpha y_0 - 3\beta y_0^2 - 2\gamma y_0^3 - 3\dot{y}_0^2 = 0.$$

The period equals $T = (2K(m)/\sqrt{w})$

In order to obtain approximate solution by means of the Galerkin method, let

$$
\begin{aligned}
y(t) &= y_0 + x(t), \\
x(0) &= 0, \\
x'(0) &= \dot{y}_0.
\end{aligned}
\tag{32}
$$

Then,

$$
\begin{aligned}
\ddot{x} + \alpha + \beta y_0 + \gamma y_0^2 + (\beta + 2\gamma y_0)x(t) + \gamma x(t)^2 &= 0, \\
x(0) &= 0, \\
x'(0) &= \dot{y}_0.
\end{aligned}
\tag{33}
$$

Assume the ansatz

$$
x = x(t) = \sum_{k=1}^{n} c_k \varphi_k(t).
\tag{34}
$$

Define $c_{-1} = c_0 = 0, c_1 = T/n\dot{x}_0$ and

$$
R_j = \int_0^T \left(\ddot{x} + \alpha + \beta y_0 + \gamma y_0^2 + (\beta + 2\gamma y_0)x(t) + \gamma x(t)^2 \right) \varphi_j(t)\mathrm{d}t.
\tag{35}
$$

Then,

$$
R_j = \frac{8c_j\left(-3n^2 + \beta T^2 + 2\gamma T^2 y_0\right) + 2c_{j-1}\left(\gamma T^2\left(c_j + 2y_0\right) + 6n^2 + \beta T^2\right) + \gamma T^2 c_{j-1}^2 + 6\gamma T^2 c_j^2 + 12T^2\left(\alpha + y_0\left(\beta + \gamma y_0\right)\right)}{12nT}
$$

$$
+ \frac{\left(\gamma T^2\left(c_j + 2y_0\right) + 6n^2 + \beta T^2\right)}{6nT}c_{j+1} + \frac{\gamma T}{12n}c_{j+1}^2 = 0.
\tag{36}
$$

Observe that the equation $R_j = 0$ is a quadratic equation in $z = c_{j+1}$. It is clear that the system $R_0 = R_1 = \ldots = R_{n-1} = 0$ may be solved recursively. We first find c_2 letting $j = 1$ in (29):

$$
c_2 = \frac{-6n^2 - T^2(\beta + 2\gamma y_0) \pm \sqrt{\left(6n^2 + \beta T^2\right)^2 + \gamma T^2\left(-2nT\dot{y}_0\left(6n^2 + \beta T^2 + 2\gamma T^2 y_0\right) - 8n^2 y_0\left(-3n^2 + \beta T^2 + \gamma T^2 y_0\right) - \gamma T^4 \dot{y}_0^2\right)/n^2 - 12\alpha\gamma T^4}}{\gamma T^2}.
\tag{37}
$$

We choose the value of c_2 that is closest to $c_1 = T/n\dot{x}_0$. Next, we set $j = 2$ in (37) and then we will find $z = c_3$. We choose the closest to c_2 solution to the quadratic equation in (29). We continue this procedure and then we will find all values of c_j ($j = 1, 2, 3, \ldots n$). Since $x(\xi_j) = x(T/nj) = c_j$, all pairs (ξ_j, c_j) will lie on the graph of the solution for sufficiently large n. Plotting these points, we obtain the graph of the solution.

On the other hand, if n is large enough, the values of c_j and c_{j+1} will be close to each other. We may use the following approximate expression for c_{j+1} in terms of c_{j-1} and c_j:

$$
c_{j+1} \approx -\frac{8c_j\left(-3n^2 + \beta T^2 + 2\gamma T^2 y_0\right) + 2c_{j-1}\left(\gamma T^2\left(c_j + 2y_0\right) + 6n^2 + \beta T^2\right) + \gamma T^2 c_{j-1}^2 + 5\gamma T^2 c_j^2 + 12T^2\left(\alpha + y_0\left(\beta + \gamma y_0\right)\right)}{2\left(2\gamma T^2\left(c_j + y_0\right) + 6n^2 + \beta T^2\right)}.
\tag{38}
$$

Example 2. Let

$$
\ddot{y} + 2y(t) - y^2(t) = 0 \wedge y(0) = 0 \wedge y'(0) = 1.
\tag{39}
$$

The exact solution is obtained from (25) and (26):

$$
y(t) = -0.641784 + \frac{(0.801114 cn(0.574208t| - 0.744849) + 1.08694 dn(0.574208t| - 0.744849)sn(0.574208t| - 0.744849)^2}{\left(1 + 0.420436 sn(0.574208t| - 0.744849)^2\right)^2}.
\tag{40}
$$

The solution is periodic and its period equals $T = 4.737423705838371$ (see Figure 3).

3.3. Duffing–Helmholtz Oscillator. Let

$$\ddot{q} + \alpha + \beta q + \gamma q^2 + \delta q^3 = 0,$$
$$q(0) = q_0, \tag{41}$$
$$q'(0) = \dot{q}_0.$$

The exact solution to i.v.p. (34) is given by

$$q(t) = A + \frac{B}{1 + C\wp(t + t_0; g_2, g_3)}, \tag{42}$$

where

$$B = -\frac{6\left(A^3\delta + A^2\gamma + A\beta + \alpha\right)}{3A^2\delta + 2A\gamma + \beta},$$

$$C = \frac{12}{3A^2\delta + 2A\gamma + \beta},$$

$$t_0 = \pm\wp^{-1}\left(\frac{q_0 - A - B}{C(A - q_0)}; g_2, g_3\right),$$

$$g_2 = \frac{1}{12}\left(-3A^4\delta^2 - 4A^3\gamma\delta - 6A^2\beta\delta - 12A\alpha\delta - 4\alpha\gamma + \beta^2\right),$$

$$g_3 = \frac{1}{216}\left(-3A^4\delta\left(\gamma^2 - 3\beta\delta\right) - 4A^3\gamma\left(\gamma^2 - 3\beta\delta\right) + 6A^2\beta\left(3\beta\delta - \gamma^2\right) - 12A\alpha\left(\gamma^2 - 3\beta\delta\right) + 27\alpha^2\delta - 6\alpha\beta\gamma + \beta^3\right),$$

$$3\delta A^4 + 4\gamma A^3 + 6\beta A^2 + 12A\alpha - \left(12q_0\alpha + 6q_0^2\beta + 4q_0^3\gamma + 3q_0^4\delta + 6\dot{q}_0^2\right) = 0.$$

This solution is valid even if $\alpha = \gamma = 0$. The solution is periodic and its period equals

$$T = 2\int_a^\infty \frac{1}{\sqrt{4x^3 - g_2x - g_3}}, \tag{44}$$

where a is the greatest real root to the cubic $4x^3 - g_2x - g_3 = 0$. Assume that $4x^3 - g_2x - g_3 = 4(x - a)(x - b)(x - c)$. Then, the period may be evaluated using the formulas

$$\int_a^u \frac{dx}{\sqrt{(x - a)(x - b)(x - c)}} = \frac{2}{\sqrt{c - b}}F\left(\sin^{-1}\left(\sqrt{\frac{(b - c)(u - a)}{(b - a)(u - c)}}\right), \frac{b - a}{b - c}\right),$$

$$\int_a^\infty \frac{dx}{\sqrt{(x - a)(x - b)(x - c)}} = \lim_{u \to \infty} \frac{2}{\sqrt{c - b}}F\left(\sin^{-1}\left(\sqrt{\frac{(b - c)(u - a)}{(b - a)(u - c)}}\right), \frac{b - a}{b - c}\right) = \frac{2}{\sqrt{c - b}}K\left(\frac{b - a}{b - c}\right),$$

$$\int_a^u \frac{dx}{\sqrt{(x - a)(x - b)(x - \overline{b})}} = \frac{F}{\sqrt[4]{(a - \alpha)^2 + \beta^2}}\left(\cos^{-1}\left(\frac{a - u + \sqrt{(a - \alpha)^2 + \beta^2}}{-a + u + \sqrt{(a - \alpha)^2 + \beta^2}}\right), \frac{\alpha - a + \sqrt{(a - \alpha)^2 + \beta^2}}{2\sqrt{(a - \alpha)^2 + \beta^2}}\right), \tag{45}$$

$$\int_a^\infty \frac{dx}{\sqrt{(x - a)(x - b)(x - \overline{b})}} = \lim_{u \to \infty}\int_a^u \frac{dx}{\sqrt{(x - a)(x - b)(x - \overline{b})}} = \frac{2}{\sqrt[4]{(a - \alpha)^2 + \beta^2}}K\left(\frac{-a + \alpha + \sqrt{(a - \alpha)^2 + \beta^2}}{2\sqrt{(a - \alpha)^2 + \beta^2}}\right),$$

$$\alpha = \text{Re}(b),$$
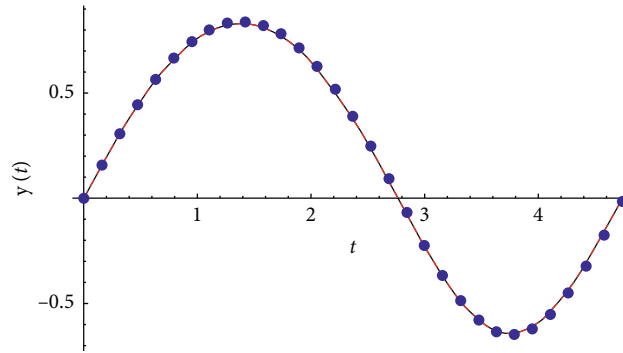
$$\beta = \text{Im}(b).$$

Figure 3: Comparison between the exact solution and the finite element method for $n = 30$.

Another expression for the exact solution is given by   where

$$q(t) = A + \frac{B}{1 + v_0 cn\left(\sqrt{a + bv_0^2}\, t,\, bv_0^2/2\left(a + bv_0^2\right)\right)}, \qquad (46)$$

$$v_0 = \frac{q_0 - A - B}{A - q_0},$$

$$3\delta A^4 + 4\gamma A^3 + 6\beta A^2 + 12\alpha A - \left(12q_0\alpha + 6q_0^2\beta + 4q_0^3\gamma + 3q_0^4\delta\right) = 0,$$

$$A^4\left(-\left(\gamma^2 - 3\beta\delta\right)\right) - 2A^3\left(\beta\gamma - 9\alpha\delta\right) - 3A^2\left(12q_0\alpha\delta + 6q_0^2\beta\delta + 4q_0^3\gamma\delta + 3q_0^4\delta^2 - 2\alpha\gamma + \beta^2\right)-$$

$$a = \frac{2A\left(12q_0\alpha\gamma + 6q_0^2\beta\gamma + 4q_0^3\gamma^2 + 3q_0^4\gamma\delta + 3\alpha\beta\right) - 12q_0\alpha\beta - 6q_0^2\beta^2 - 4q_0^3\beta\gamma - 3q_0^4\beta\delta - 9\alpha^2}{\left(A - q_0\right)\left(3A^3\delta + A^2\left(3q_0\delta + 4\gamma\right) + A\left(4q_0\gamma + 3q_0^2\delta + 6\beta\right) + 6q_0\beta + 4q_0^2\gamma + 3q_0^3\delta + 12\alpha\right)}, \qquad (47)$$

$$b = -\frac{3\left(A^3\delta + A^2\gamma + A\beta + \alpha\right)^2}{\left(A - q_0\right)\left(3A^3\delta + A^2\left(3q_0\delta + 4\gamma\right) + A\left(4q_0\gamma + 3q_0^2\delta + 6\beta\right) + 6q_0\beta + 4q_0^2\gamma + 3q_0^3\delta + 12\alpha\right)},$$

$$B = -\frac{\left(A - q_0\right)\left(3A^3\delta + A^2\left(3q_0\delta + 4\gamma\right) + A\left(4q_0\gamma + 3q_0^2\delta + 6\beta\right) + 6q_0\beta + 4q_0^2\gamma + 3q_0^3\delta + 12\alpha\right)}{3\left(A^3\delta + A^2\gamma + A\beta + \alpha\right)}.$$

Let $q_0 = 0$. Assume the ansatz   Define $c_0 = c_{n+1} = 0$. We have

$$q = q(t) = \sum_{k=1}^{n} c_k \varphi_k(t). \qquad (48)$$

$$\int_0^T \left(\left(\ddot{q}t\right) + \alpha + \beta q(t) + \gamma q^2(t) + \delta q^3(t)\right)\varphi_j(t)\mathrm{d}t = \frac{c_{j-1} - 2c_j + c_{j+1}}{h}$$

$$+ \alpha h + \frac{1}{6}\beta h\left(c_{j-1} + 4c_j + c_{j+1}\right)$$

$$+ \frac{1}{12}\gamma h\left(c_{j-1}^2 + 2c_j c_{j-1} + 6c_j^2 + c_{j+1}^2 + 2c_j c_{j+1}\right) \qquad (49)$$

$$+ \frac{1}{20}\delta h\left(c_{j-1}^3 + 2c_j c_{j-1}^2 + 3c_j^2 c_{j-1} + 8c_j^3 + c_{j+1}^3 + 2c_j c_{j+1}^2 + 3c_j^2 c_{j+1}\right)$$

$$\text{for} \quad j = 1, 2, 3, \ldots .n.$$

This gives an algebraic system of nonlinear equations. The initial data are

$$c_0 = 0,$$

$$c_1 = c_1 = \frac{T\dot{q}_0}{n}. \tag{50}$$

$$\frac{1}{20}\delta h z^3 + \frac{1}{60}h\left(6\delta c_j + 5\gamma\right)z^2 + \frac{h^2\left(10\beta + 10\gamma c_j + 9\delta c_j^2\right) + 60}{60h}$$

$$z + \frac{1}{60h}\left(\begin{array}{c} c_{j-1}\left(h^2 c_j\left(9\delta c_j + 10\gamma\right) + 10\left(\beta h^2 + 6\right)\right) + 40\beta h^2 c_j + \\ h^2 c_{j-1}^2\left(6\delta c_j + 5\gamma\right) + 6h^2 c_j^2\left(4\delta c_j + 5\gamma\right) + 3\delta h^2 c_{j-1}^3 - 120c_j + 60\alpha h^2 \end{array}\right) = 0, \tag{51}$$

where

$$z = c_{j+1}.$$

Example 3. Let $\alpha = 1$, $\beta = -1$, $\gamma = 1$, $\delta = 1$, $q_0 = 0$, $\dot{q}_0 = 0.3$, and $T = 20$ (see Figure 4 for a comparison between the exact solution and the Galerkin method for different values of n).

3.4. Duffing Equation with Damping and Forcing. Let

$$\ddot{x} + 2\varepsilon\dot{x} + \alpha x + \beta x^3 = F\cos(\omega t),$$

$$x(0) = 0,$$

$$x'(0) = \dot{x}_0, \tag{52}$$

$$0 \le t \le T.$$

Observe that algebraic system (39) may be solved *recursively* using the Tartaglia formula for the cubic. Indeed, we may write

Assume the ansatz

$$x = x(t) = \sum_{k=1}^{n} c_k \varphi_k(t). \tag{53}$$

Define $c_0 = c_{n+1} = 0$. We have

$$\int_0^T \left(\ddot{x} + 2\varepsilon\dot{x} + \alpha x + \beta x^3 - F\cos(\omega t)\right)\varphi_j(t)dt$$

$$= \frac{1}{20}\beta h c_{j+1}^3 + \frac{1}{10}\beta h c_j c_{j+1}^2 + \left(\frac{3}{20}\beta h c_j^2 + \varepsilon + \frac{\alpha h}{6} + \frac{1}{h}\right)$$

$$c_{j+1} + \frac{1}{60h}\left(\begin{array}{c} 8\left(5c_j\left(\alpha h^2 - 3\right) + 3\beta h^2 c_j^3 - \frac{30F}{\omega^2}\sin^2\left(\frac{h\omega}{2}\right)\cos(hj\omega)\right) + \\ c_{j-1}\left(9\beta h^2 c_j^2 + 10\left(\alpha h^2 - 6\varepsilon h + 6\right)\right) + 3\beta h^2 c_{j-1}^3 + 6\beta h^2 c_j c_{j-1}^2 \end{array}\right) = 0, \tag{54}$$
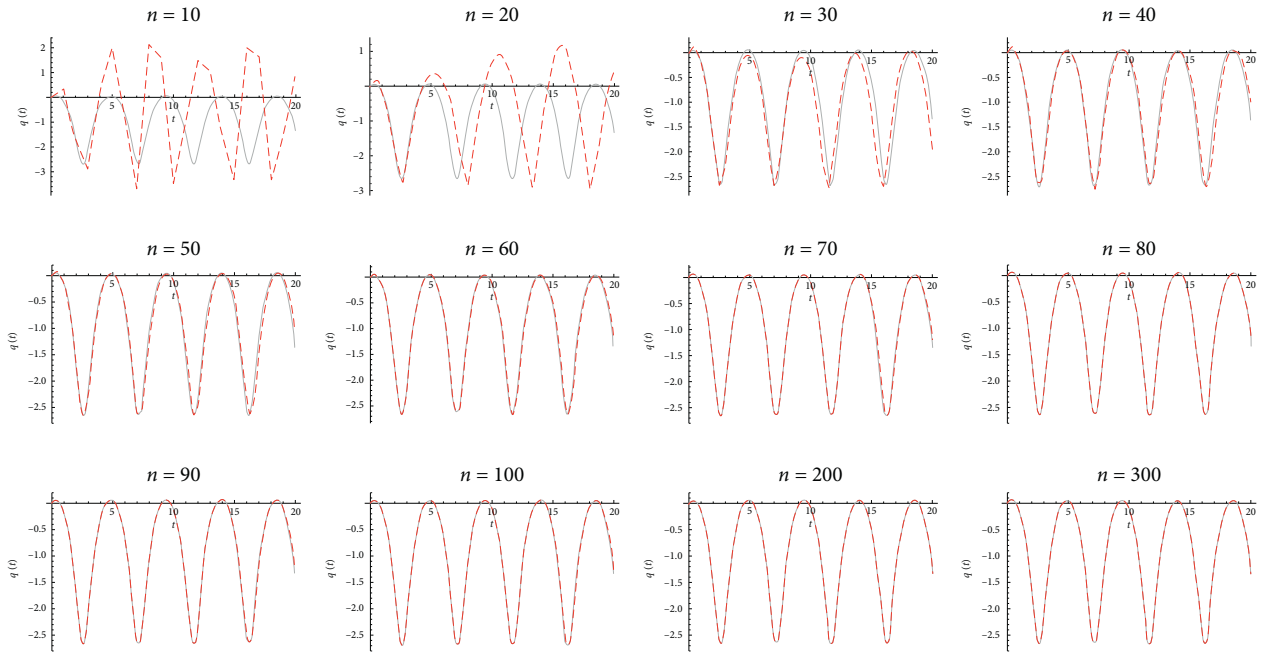
$$j = 1, 2, 3, \ldots.$$

FIGURE 4: Galerkin hat method for different values of n.

We may solve this system recursively using the Tartaglia formula for the cubic. The initial data are

$$c_0 = 0,$$

$$c_1 = \frac{T}{n}\dot{x}_0. \tag{55}$$

3.5. Forced Van der Pol–Duffing Equation. Let

$$\ddot{x} - \varepsilon\left(1 - x^2\right)\dot{x} + \alpha x + \beta x^3 = F\cos(\omega t),$$

$$x(0) = 0,$$

$$x'(0) = \dot{x}_0, \tag{56}$$

$$0 \leq t \leq T.$$

Assume the ansatz

$$x = x(t) = \sum_{k=1}^{n} c_k \varphi_k(t). \tag{57}$$

Define $c_0 = c_{n+1} = 0$. We have

$$\int_0^T \left(\ddot{x} - \varepsilon\left(1 - x^2\right)\dot{x} + \alpha x + \beta x^3 - F\cos(\omega t)\right)$$

$$\varphi_j(t)\mathrm{d}t = \frac{1}{60}(5\varepsilon + 3\beta h)c_{j+1}^3 + \frac{1}{60}c_j(5\varepsilon + 6\beta h)c_{j+1}^2 + \left(\frac{1}{60}c_j^2(5\varepsilon + 9\beta h) - \frac{\varepsilon}{2} + \frac{\alpha h}{6} + \frac{1}{h}\right)$$

$$c_{j+1} + \frac{1}{60h}\left(\begin{array}{c} 8\left(5c_j\left(\alpha h^2 - 3\right) + 3\beta h^2 c_j^3 - \dfrac{30F}{\omega^2}\sin^2\left(\dfrac{h\omega}{2}\right)\cos(hj\omega)\right) + \\[2mm] c_{j-1}\left(hc_j^2(9\beta h - 5\varepsilon) + 10\left(\alpha h^2 + 3\varepsilon h + 6\right)\right) + hc_{j-1}^3(3\beta h - 5\varepsilon) + hc_j c_{j-1}^2(6\beta h - 5\varepsilon) \end{array}\right) = 0, \quad \text{for } j = 1, 2, 3, \ldots.$$

$$\tag{58}$$

We may solve this system recursively using the Tartaglia formula for the cubic. The initial data are

$$c_0 = 0,$$
$$c_1 = \frac{n\dot{x}_0}{T}. \tag{59}$$

3.6. Conservative Nonlinear Oscillators. Conservative single-degree-of-freedom nonlinear oscillators are modelled by second-order autonomous ordinary differential equations of the form

$$\ddot{x} + F(x) = 0,$$
$$x(0) = x_0, \tag{60}$$
$$x'(0) = 0.$$

Using Chebyshev polynomials or minimization techniques, we may approximate the function $F(x)$ by means of a cubic polynomial, say $F(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$, and this allows us to study i.v.p. (44) using the solution to the i.v.p.

$$\ddot{x} + \alpha + \beta x + \gamma x^2 + \delta x^3 = 0,$$
$$x(0) = x_0, \tag{61}$$
$$x'(0) = 0.$$

For example, let us consider a mass attached to two stretched elastic springs [8]. For this problem, the function $F(x)$ has the form

$$F(x) = x - \frac{\lambda x}{\sqrt{1 + x^2}}, \quad 0 < \lambda \leq 1. \tag{62}$$

Suppose that $|x| \leq A$. We have $F(x) \approx px + qx^3$, where

$$p = 1 + \frac{15\lambda\left(\left(20A^2 + 21\right)\sinh^{-1}(A) - 3A\sqrt{A^2 + 1}\left(2A^2 + 7\right)\right)}{32A^5},$$

$$q = \frac{35\lambda\left(A\sqrt{A^2 + 1}\left(2A^2 + 15\right) - 3\left(4A^2 + 5\right)\sinh^{-1}(A)\right)}{32A^7}. \tag{63}$$

Then, the problem reduces to that of solving a Duffing equation.

4. Analysis and Discussion

We have described the way to solve strongly nonlinear oscillators by means of the Galerkin method. In general, any second-order ordinary differential equation may be solved using the Runge–Kutta numerical method. In general, the Runge-Kutta numerical solution offers a more accurate than the Galerkin solution. On the other hand, the Galerkin method offers the possibility to write the solution as a linear combination of hat functions. In this sense, the Galerkin method is a kind of analytical method. For a given conservative oscillator (45), we may approximate the function $F(x)$ by a cubic polynomial and then we replace the original problem with problem (45), which has exact analytical solution. However, the exact solution demands the evaluation of either a Jacobian or elliptic Weierstrass function, which has extra costs.

5. Conclusions

The Galerkin method offers a way to obtain semianalytical solution to a given nonlinear oscillator of special form (1). The exact solution to it is not known in general. There are other ways to solve it. Perturbative methods like the Lindstedt–Poincaré method or the Krylov–Bogoliubov–Mitropolsky method are also possible for this end. The advantage of the proposed method consists of the possibility to solve the Galerkin equations iteratively using an algebraic equation. The Galerkin hat method is usually used to solve second-order linear ODEs. In this work, we extended it to a class of nonlinear oscillators. Other methods for solving nonlinear differential equations may be found in [8–10].

Appendix

Formulas for Computing

$$
\left(
\begin{array}{c}
c_{j-1}n^{\rho}\left((j-1)^2(2j+\rho+1)((j-1)T)^{\rho}-j^2(2j-\rho-3)(jT)^{\rho}\right)+ \\[4pt]
c_{j+1}n^{\rho}\left(j^2(2j+\rho+3)(jT)^{\rho}+(j+1)^2(-2j+\rho+1)((j+1)T)^{\rho}\right)+ \\[4pt]
2c_j\left(\dfrac{j\left(-2j(\rho+3)n^{\rho}+(-(j-3)j-3)\left(\frac{(j-1)n}{j}\right)^{\rho}+(j(j+3)+3)\left(\frac{n}{j}+n\right)^{\rho}\right)+(jT)^{\rho}}{\left(\frac{(j-1)n}{j}\right)^{\rho}+\left(\frac{n}{j}+n\right)^{\rho}}\right)
\end{array}
\right),
$$

$$
\int_0^T t^{\rho} x(t)^2 \varphi_j(t)\,dt = \frac{T}{(\rho+1)(\rho+2)(\rho+3)(\rho+4)n^{2\rho+1}}
$$

$$
\left(\left(\left(
\begin{array}{c}
2c_{j-1}c_j n^{\rho}\left(\dfrac{\left(j^2(6j^2-4j(\rho+4)+(\rho+3)(\rho+4))(jT)^{\rho}-\right)}{2(j-1)^3(3j+\rho+1)((j-1)T)^{\rho}}\right)+ \\[6pt]
2c_j c_{j+1}n^{\rho}\left(\dfrac{j^2(6j^2+4j(\rho+4)+(\rho+3)(\rho+4))(jT)^{\rho}+}{2(j+1)^3(-3j+\rho+1)((j+1)T)^{\rho}}\right)+ \\[6pt]
6c_j^2\left(j\left(j\left(\dfrac{-(2j^2+(\rho+3)(\rho+4))n^{\rho}+((j-4)j+6)\left(\frac{(j-1)n}{j}\right)^{\rho}+}{(j(j+4)+6)\left(\frac{n}{j}+n\right)^{\rho}}\right)+4\left(\left(\left(\frac{1}{j}+1\right)n\right)^{\rho}-\left(\frac{(j-1)n}{j}\right)^{\rho}\right)\right)+\left(\left(\frac{1}{j}+1\right)n\right)^{\rho}+\left(\frac{(j-1)n}{j}\right)^{\rho}\right)+ \\[6pt]
(jT)^{\rho}+c_{j-1}^2 n^{\rho}\left((j-1)^2\left(6j^2+4j(\rho+1)+(\rho+1)(\rho+2)\right)((j-1)T)^{\rho}-2j^3(3j-\rho-4)(jT)^{\rho}\right)+ \\[6pt]
c_{j+1}^2 n^{\rho}\left((j+1)^2\left(6j^2-4j(\rho+1)+(\rho+1)(\rho+2)\right)((j+1)T)^{\rho}-2j^3(3j+\rho+4)(jT)^{\rho}\right)
\end{array}
\right)\right)\right),
$$

$$
\int_0^T t^{\rho} x(t)^3 \varphi_j(t)\,dt = \frac{T}{(\rho+1)(\rho+2)(\rho+3)(\rho+4)(\rho+5)n^{2\rho+1}}
$$

$$
\left(
\begin{array}{c}
8c_j^3\left(j\left(\dfrac{-j(\rho+5)\left(6j^2+(\rho+3)(\rho+4)\right)n^{\rho}+}{\begin{array}{c}3(j(j(j+5)+10)+10)+5)\left(\left(\frac{1}{j}+1\right)n\right)^{\rho}-\\ 3(j(j((j-5)j+10)-10)+5)\left(\frac{(j-1)n}{j}\right)^{\rho}\end{array}}\right)+3\left(\left(\left(\frac{1}{j}+1\right)n\right)^{\rho}+\left(\frac{(j-1)n}{j}\right)^{\rho}\right)\right) \\[10pt]
(jT)^{\rho}+3c_{j-1}c_j^2 n^{\rho}\left(\dfrac{6(j-1)^4(4j+\rho+1)((j-1)T)^{\rho}-}{j^2\left(\frac{24j^3-18j^2(\rho+5)+}{6j(\rho+4)(\rho+5)-(\rho+3)(\rho+4)(\rho+5)}\right)(jT)^{\rho}}\right)+ \\[10pt]
6c_j c_{j+1}^2 n^{\rho}\left(\dfrac{(j+1)^3\left(12j^2-6j(\rho+1)+(\rho+1)(\rho+2)\right)((j+1)T)^{\rho}-}{j^3\left(\frac{12j^2+6j(\rho+5)+}{(\rho+4)(\rho+5)}\right)(jT)^{\rho}}\right)+ \\[10pt]
6c_{j-1}^2 c_j n^{\rho}\left(\dfrac{j^3\left(12j^2-6j(\rho+5)+(\rho+4)(\rho+5)\right)(jT)^{\rho}-}{(j-1)^3\left(\frac{12j^2+6j(\rho+1)+}{(\rho+1)(\rho+2)}\right)((j-1)T)^{\rho}}\right)+ \\[10pt]
3c_j^2 c_{j+1} n^{\rho}\left(\dfrac{j^2\left(24j^3+18j^2(\rho+5)+6j(\rho+4)(\rho+5)+(\rho+3)(\rho+4)(\rho+5)\right)(jT)^{\rho}+}{6(j+1)^4(-4j+\rho+1)((j+1)T)^{\rho}}\right)+ \\[10pt]
c_{j-1}^3 n^{\rho}\left((j-1)^2\left(\frac{24j^3+18j^2(\rho+1)+6j(\rho+1)(\rho+2)+}{(\rho+1)(\rho+2)(\rho+3)}\right)((j-1)T)^{\rho}-6j^4(4j-\rho-5)(jT)^{\rho}\right)+ \\[10pt]
c_{j+1}^3 n^{\rho}\left(6j^4(4j+\rho+5)(jT)^{\rho}+(j+1)^2\left(\frac{-24j^3+18j^2(\rho+1)-}{6j(\rho+1)(\rho+2)+(\rho+1)(\rho+2)(\rho+3)}\right)((j+1)T)^{\rho}\right)
\end{array}
\right),
$$

$$\int_0^T \sin(\omega t) x(t) \varphi_j(t) dt$$

$$= \frac{-4c_j(\sin(h\omega) - h\omega)\sin(hj\omega) + c_{j-1}(-h\omega\sin(h(j-1)\omega) + \sin(hj\omega)) + 2\cos(h(j-1)\omega) - 2\cos(hj\omega)) + c_{j+1}(-h\omega(\sin(hj\omega) + \sin(h(j+1\omega))) + 2\cos(hj\omega) - 2\cos(h(j+1)\omega))}{h^2\omega^3},$$

$$\int_0^T \cos(\omega t) x(t) \varphi_j(t) dt = -\frac{c_{j-1}(2(\sin(h(j-1)\omega) - \sin(hj\omega) + h\omega\cos(h(j-1)\omega) + \cos(hj\omega))) + 4c_j(\sin(h\omega) - h\omega)\cos(hj\omega) + c_{j+1}(2(\sin(hj\omega) - \sin(h(j+1)\omega)) + h\omega(\cos(hj\omega) + \cos(h(j+1)\omega)))}{h^2\omega^3}$$

$$\int_0^T \exp(x(t)) \varphi_j(t) dt = \frac{T^2\left((e^{c_j} - e^{c_{j+1}})c_{j-1} + (e^{c_{j-1}} - 2e^{c_j} + e^{c_{j+1}})c_j - (e^{c_{j-1}} - e^{c_j})c_{j+1}\right)}{n^2(c_{j-1} - c_j)(c_j - c_{j+1})},$$

$$\int_0^T x(t) x'(t) \varphi_j(t) dt = -\frac{1}{6}(c_{j-1} - c_{j+1})(c_{j-1} + c_j + c_{j+1}),$$

$$\int_0^T x(t)^2 x'(t) \varphi_j(t) dt = \frac{1}{12}\left(c_{j+1}^3 + c_j c_{j+1}^2 + c_j^2 c_{j+1} - c_{j-1}(c_{j-1}^2 + c_j c_{j-1} + c_j^2)\right),$$

$$\int_0^T x(t)^3 x'(t) \varphi_j(t) dt = \frac{1}{20}\left(c_{j+1}^4 + c_j c_{j+1}^3 + c_j^2 c_{j+1}^2 + c_j^3 c_{j+1} - c_{j-1}(c_{j-1} + c_j)(c_{j-1}^2 + c_j^2)\right),$$

$$\int_0^T x(t) x'(t)^2 \varphi_j(t) dt = \frac{n\left(c_{j-1}^3 - 3c_j^2 c_{j-1} + 4c_j^3 + c_{j+1}^3 - 3c_j^2 c_{j+1}\right)}{6T},$$

$$\int_0^T x(t)^2 x'(t)^2 \varphi_j(t) dt = \frac{n\left(c_{j-1}^4 - 4c_j^3 c_{j-1} + 6c_j^4 + c_{j+1}^4 - 4c_j^3 c_{j+1}\right)}{12T},$$

$$\int_0^T x(t)^3 x'(t)^2 \varphi_j(t) dt = \frac{1}{20}\left(c_{j+1}^4 + c_j c_{j+1}^3 + c_j^2 c_{j+1}^2 + c_j^3 c_{j+1} - c_{j-1}(c_{j-1} + c_j)(c_{j-1}^2 + c_j^2)\right),$$

$$\int_0^T x(t)^3 x'(t) \varphi_j(t) dt = \frac{n\left(c_{j-1}^5 - 5c_j^4 c_{j-1} + 8c_j^5 + c_{j+1}^5 - 5c_j^4 c_{j+1}\right)}{20T},$$

$$\int_0^T x(t)^4 x'(t) \varphi_j(t) dt = \frac{1}{30}\left(c_{j+1}^5 + c_j c_{j+1}^4 + c_j^2 c_{j+1}^3 + c_j^3 c_{j+1}^2 + c_j^4 c_{j+1} - c_{j-1}(c_{j-1}^4 + c_j c_{j-1}^3 + c_j^2 c_{j-1}^2 + c_j^3 c_{j-1} + c_j^4)\right)$$

$$\int_0^T x(t)^5 x'(t) \varphi_j(t) dt = \frac{1}{42}\left(\begin{array}{c} c_{j+1}^6 + c_j c_{j+1}^5 + c_j^2 c_{j+1}^4 + c_j^3 c_{j+1}^3 + c_j^4 c_{j+1}^2 + c_j^5 c_{j+1} - \\ c_{j-1}(c_{j-1} + c_j)(c_{j-1}^2 - c_j c_{j-1} + c_j^2)(c_{j-1}^2 + c_j c_{j-1} + c_j^2) \end{array}\right)$$

$$\int_0^T \varphi_j(t)\cos(t\omega) dt = \frac{2n(1 - \cos(T\omega/n))\cos(jT\omega/n)}{T\omega^2},$$

$$\int_0^T x(t)^5 x'(t) \varphi_j(t) dt = \frac{1}{42}\left(\begin{array}{c} c_{j+1}^6 + c_j c_{j+1}^5 + c_j^2 c_{j+1}^4 + c_j^3 c_{j+1}^3 + c_j^4 c_{j+1}^2 + c_j^5 c_{j+1} - \\ c_{j-1}(c_{j-1} + c_j)(c_{j-1}^2 - c_j c_{j-1} + c_j^2)(c_{j-1}^2 + c_j c_{j-1} + c_j^2) \end{array}\right),$$

$$\int_0^T \varphi_j(t)\cos(t\omega) dt = \frac{2n(1 - \cos(T\omega/n))\cos(jT\omega/n)}{T\omega^2}.$$

(A.1)

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

[1] E. Gluskin, "A nonlinear resistor and nonlinear inductor using a nonlinear capacitor," *Journal of the Franklin Institute*, vol. 336, no. 7, pp. 1035–1047, 1999.

[2] S. Morfa and J. C. Comte, "A nonlinear oscilators netwok devoted to image processing," *International Journal of Bifurcation and Chaos*, vol. 14, pp. 1385–1394, 2009.

[3] W. Lestari and S. Hanagud, "Nonlinear vibration of buckled beams: some exact solutions," *International Journal of Solids and Structures*, vol. 38, pp. 4741–4757, 2001.

[4] V. S. Aslanov, "The oscillations of a body with an orbital tethered system," *Journal of Applied Mathematics and Mechanics*, vol. 71, no. 6, pp. 926–932, 2007.

[5] J. E. C. Hernández, A. H. Salas, and J. G. E. Lugo, "Exact solutions for a nonlinear model," *Applied Mathematics and Computation*, vol. 217, pp. 1646–1651, 2010.

[6] Y. F. Liu and G. R. Li, "Matter wave soliton solutions of the cubic–quintic nonlinear Schrödinger equation with an

anharmonic potential," *Applied Mathematics and Computation*, vol. 219, no. 9, pp. 4847–4852, 2013.

[7] P. J. Jolmes, "A nonlinear oscillator with a strange attractor," *Proceedings of the Royal Society*, vol. 292, pp. 419–448, 1979.

[8] N. Nayfeth and D. T. Mook, *Non-linear Oscillations*, John Wiley, Hoboken, NJ, USA, 1973.

[9] J. A. Almendral, "Integrability and symmetries for the helmholtz oscillator with friction," *Journal of physics a general physics*, vol. 36, no. 3, p. 695, 2003.

[10] A. H. Salas and J. E. Castillo, "Exact solutions to cubic Duffing equation for a nonlinear electrical circuit," *Electronic Vision-More than Just a Solid State*, vol. 8, no. 1, 2014.