

Research Article

Reliability Evaluation of Generalized Exchanged X -Cubes Based on the Condition of g -Good-Neighbor

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In the cloud computing environment with massive information services and decision-making resources, the accuracy and reliability of information are more important than previous single closed systems. Therefore, ensuring the reliability of information and the stable operation of the system are the core problems in the research fields such as the Internet Plus and the Internet of Things. The connectivity and diagnosability are two important measures for the fault tolerance of multiprocessor systems. The g -good-neighbor conditional connectivity (R^g -connectivity) is the minimum number of nodes that make the graph disconnected, and each node has at least g neighbors in every remaining component. The g -good-neighbor conditional diagnosability (g -GNCD) is the maximum number of faulty processors that has been correctly identified in a system, and any fault-free processor has no less than g fault-free neighbors. Exchanged X -cubes are a class of irregular networks, obtained by deleting links from hypercubes and some variant networks of hypercubes (X -cubes). They not only combine the advantages of X -cubes but also reduce the interconnection complexity. Exchanged X -cubes classify its nodes into two different classes clusters with a unique connecting rule. In this paper, we propose the generalized exchanged X -cubes framework so that architecture can be constructed by different connecting rules. Furthermore, we study the R^g -connectivity and g -GNCD of generalized exchanged X -cubes under the PMC and MM* models. As applications, the R^g -connectivity and g -GNCD of generalized exchanged hypercubes, dual-cube-like networks, generalized exchanged crossed cubes, and locally generalized exchanged twisted cubes are determined, respectively.

1. Introduction

With the expansion of network scale and the improvement of complexity, the reliability and stability of the system become more and more important. How to ensure the correct and efficient operation of the system is an important research topic for wireless sensor networks and distributed systems. The distributed system disperses the computing tasks which are originally collected on one computer to polymorphic computer for parallel processing. It has many advantages, such as resource sharing, openness, concurrency, scalability, and fault tolerance. In the operation of the multiprocessor system, processor failure is inevitable. It may slow down the

communication of information or even lead to paralyze of the system, thereby affecting the normal operation of the multiprocessor system and bringing huge losses. For example, on Google and Amazon systems, the failure of processors (servers) for several hours can bring millions of dollars of losses. Therefore, fault tolerance is very important for the construction and maintenance of systems [1, 2]. In fact, a multiprocessor system can be usually enlightened as a simple connected graph, where each processor represents a node of the graph, and each link between two processors represents an edge between two nodes in the graph. The graph is called the interconnection network of this multiprocessor system. Thus, some parameters of a graph as an interconnection

network can be used to measure the reliability of a multi-processor system. In the following, we do not distinguish among multiprocessor systems, interconnection networks, and graphs.

An important evaluating parameter for the fault tolerance of a system (modeled by graph G), the *connectivity*, is denoted by $\kappa(G)$, which is the minimum number of nodes that make the graph disconnected. So far, connectivities for many famous networks have been proven. Nevertheless, there is a shortcoming for using traditional connectivity as a parameter of fault tolerance, which is considered a highly unlikely phenomenon in reality that all nodes adjacent to a node have failed simultaneously. Therefore, Esfahanian and Hakimi [3] proposed a new measure to overcome this shortcoming, the *restricted connectivity*, which limits that all adjacent nodes of any node cannot fail at the same time. Later, a generalized restricted connectivity concept, the *g -restricted connectivity (R^g -connectivity)*, was proposed by Latifi et al. [4], which defines that each node of any remaining component after deleting all faulty nodes has degree at least g . In recent years, they have attracted much interest of theoretical computer scientists and mathematicians. Xu et al. [5] determined the R^g -connectivity of hierarchical cubic networks and complete cubic networks. Ning [6] studied the R^g -connectivity of exchanged crossed cubes. Yuan et al. [7] explored the R^g -connectivity of κ -ary n -cube networks. Lin et al. [8] obtained the R^g -connectivity of (n, k) -arrangement graphs.

Identifying all faulty processors in a multiprocessor system (in brief, system) is called *system-level diagnosis*. A system is t -diagnosable when all faulty processors can be detected, provided that the number of faulty processors in it does not exceed t . The maximum number of faulty processors that the system can precisely point out is as known as the *diagnosability* of the system. In *system-level diagnosis*, there are several well-known models.

The *PMC model* is the first model, proposed by Preparata et al. [9], which is a test-based model, assumes that the adjacent processors can perform tests on each other. For any adjacent processors in a system, the ordered pair $\langle x, y \rangle$ is called a test that x diagnoses its neighbor y , where x is a tester and y is a testee. In case x diagnoses y to be faulty (resp., fault-free), the outcome of the test $\langle x, y \rangle$ is 1 (resp., 0). Moreover, the outcome is reliable in the present of the tester x is fault-free. Another model, the *MM model*, was proposed by Maeng and Malek [10], which is a comparison-based model. In MM model, a comparator processor z sends the same test to its two neighbors x, y (i.e., comparison nodes) and then compares their responses. Let a labeled edge $(x, y)_z$ be a comparison performed that two processors x and y are compared by a processor z , where x is adjacent to z and y is also adjacent to z . If the comparator processor z is fault-free, and the responses of x and y are identical, then both comparison processors x and y are fault-free; on the other hand, if the responses of x and y are different, then at least one of x, y is faulty. Furthermore, if both comparison processors x and y are faulty, the responses of x and y are distinct. In addition, the comparison $(x, y)_z$ is unreliable in the present if the comparator node z is faulty. The *MM* model* (proposed by Sengupta and Dahbura) [11] is a special MM model, which is

assumed that each processor must compare each pair of its adjacent processors.

Since there is no restrictive condition on the distribution pattern of faulty processors, the classical diagnosability of a system is quite small. In order to increase the diagnosability, Lai et al. [12] proposed a more realistic parameter of diagnosability, *conditional diagnosability*, which limited that all the neighbors of any processor cannot be faulty at the same time in a system. Recently, Peng et al. [13] proposed the notion of *g -good-neighbor conditional diagnosability (g -GNCD)*, which is the maximum number of faulty processors that can be identified under the condition that every fault-free processor has no less than g fault-free neighbors. Peng et al. [13] (resp., Wang et al. [14]) established the g -GNCD of hypercubes under the PMC model (resp., MM* model). Li et al. [15] introduced the diagnosability and 1-good-neighbor conditional diagnosability of hypercubes with missing links and broken-down nodes under the PMC model. Yuan et al. [7] studied the g -good-neighbor conditional diagnosabilities of k -ary n -cube networks under the PMC model and the MM* model. Xu et al. [5] established the g -good-neighbor conditional diagnosabilities of complete cubic networks under the PMC model and the MM* model. Lin et al. [16] evaluated the g -good-neighbor conditional diagnosabilities of (n, k) -arrangement graphs under the PMC model and the MM* model. Guo et al. [17] studied the g -good-neighbor conditional diagnosability of the crossed cubes under the PMC model and the MM* model. Li et al. [18] introduced this concept into a family of data center networks—DCell—and determined the g -good-neighbor conditional diagnosabilities of DCell under the PMC model and the MM* model.

The R^g -connectivity (or g -GNCD) of different networks are usually determined independently. It is a very worthwhile topic to explore a unified method to get them in different networks. A family of exchanged networks (i.e., exchanged X -cubes) have some common properties, so that their R^g -connectivity (or g -GNCD) can be studied by a uniform method. The family of exchanged X -cubes not only combine the advantages of hypercubes and some variant networks of hypercubes (X -cubes) but also reduce the interconnection complexity. Exchanged X -cubes classify its nodes into two different classes clusters with a unique connecting rule. In this paper, we propose the generalized exchanged X -cubes framework so that architecture can be constructed by different connecting rules. There are some of the better properties in generalized exchanged X -cubes, such as smaller diameter, fewer edges, lower cost factor, and low latency. Based on the fine properties, the network's hardware and communication costs are reduced, and a greater balance between performance and cost can be achieved. Due to the excellent properties of the generalized exchanged X -cubes, they can be used as the logical topologies in the peer-to-peer environment [19].

In recent years, the research on the relationship between the R^g -connectivity and the g -GNCD of regular networks under certain conditions has been widely developed [20–24], while this paper will study the R^g -connectivity and the g -GNCD of a class of irregular networks (i.e., generalized

exchanged X -cubes). We first establish the R^g -connectivity of generalized exchanged X -cubes. Next, we evaluate the g -GNCD of generalized exchanged X -cubes. As applications, we obtain the R^g -connectivity and g -GNCD of generalized exchanged hypercubes, dual-cube-like networks, generalized exchanged crossed cubes, and locally generalized exchanged twisted cubes.

The remainder of this paper is organized as follows. Section 2 provides the terms and notations used throughout the paper. Section 3 evaluates the R^g -connectivity of generalized exchanged X -cubes. Section 4 establishes the g -GNCD of generalized exchanged X -cubes. Section 5 gives some applications based on the results in Section 3 and Section 4. In Section 6, we illustrate the advantages of R^g -connectivity and g -GNCD compared to traditional connectivity and traditional diagnosability, respectively. Finally, we finish the whole paper by concluding in Section 7.

2. Preliminaries

2.1. Terminology and Notations. In this paper, a multiprocessor system is usually represented by a simple undirected graph (in brief, a graph). For terminology and notations not defined in this paper, we follow the reference [25]. We use $G = (V(G), E(G))$ to represent a graph, where $V(G)$ representing a nonempty and finite *node set* and $E(G) = \{(u, v) \mid (u, v) \text{ is an unordered pair of } V(G)\}$ representing an *edge set*. Two nodes u and v are *adjacent*, denoted by $(u, v) \in E(G)$. The set of *neighbors* of node u in G is denoted by $N_G(u) = \{v \in V(G) \mid (u, v) \in E(G)\}$. If $R \subseteq V(G)$, let $G[R]$ denote the subgraph of G induced by the node subset R in G . And we denote $G - R$ as $G[V(G) \setminus R]$. We set $N_G(R) = \{v \in V(G) \setminus R \mid (u, v) \in E(G) \text{ and } u \in R\} = \bigcup_{u \in R} N_G(u) \setminus R$ and $N_G[R] = N_G(R) \cup R$. Two binary strings $u = u_1u_0$ and $v = v_1v_0$ are pair related, denoted by $u \sim v$, if and only if $(u, v) \in \{(00, 00), (01, 11), (10, 10), (11, 01)\}$. The case that u and v are not pair related is denoted by $u \not\sim v$ [26].

The *degree* of u in G is denoted by $\deg_G(u) = |N_G(u)|$. Let $\delta(G) = \min \{\deg_G(u) \mid u \in V(G)\}$, $\Delta(G) = \max \{\deg_G(u) \mid u \in V(G)\}$. K_n is defined as a complete graph with n nodes. A *path* P is a sequence of distinct nodes with any two consecutive nodes in P that are adjacent. We use $G_1 \cong G_2$ to represent the graph G_1 is isomorphic to the graph G_2 . A *component* is defined as a maximally connected subgraph of a graph.

Definition 1 (see [27]). Let $R \subseteq V(G)$. R is called a *node-cut* if $G - R$ is disconnected. If there exists a node-cut R with $|R| = k$, then R is called a k -*node-cut*. The *connectivity* $\kappa(G)$ of G is defined as the minimum k such that G has a k -node-cut.

Definition 2 (see [4]). Let g be a positive integer and $R \subseteq V(G)$. If $G - R$ is disconnected and each remaining component has minimum degree at least g , then R is called an R^g -*cut*.

Definition 3 (see [4]). The R^g -connectivity of G , denoted by $\kappa^g(G)$, is the minimum cardinality over all R^g -cuts of G .

2.2. The g -Good-Neighbor Conditional Diagnosability. Under the PMC model and MM* model, we call the notation Ω as the *syndrome* of the system, which is defined as the set of all test (comparison) results in a system G , where test results are based on the PMC model and comparison results are based on the MM* model. Define a faulty set F , where $\forall i \in F$, i is a faulty processor. Let $\Omega(F)$ be the set of test (comparison) results which could be produced if F is the faulty node set. We use \widehat{F}_1 and \widehat{F}_2 to represent two distinct faulty sets of $V(G)$. In case $\Omega(\widehat{F}_1) \cap \Omega(\widehat{F}_2) = \emptyset$, we call these two distinct faulty sets \widehat{F}_1 and \widehat{F}_2 *distinguishable*, and $(\widehat{F}_1, \widehat{F}_2)$ a *distinguishable pair*; otherwise, \widehat{F}_1 and \widehat{F}_2 are *indistinguishable*, and $(\widehat{F}_1, \widehat{F}_2)$ is an *indistinguishable pair*. Let $\widehat{F}_1 \Delta \widehat{F}_2$ be the symmetric difference $(\widehat{F}_1 - \widehat{F}_2) \cup (\widehat{F}_2 - \widehat{F}_1)$ between \widehat{F}_1 and \widehat{F}_2 . In [28], under the PMC model, the sufficient and necessary condition for two different subsets \widehat{F}_1 and \widehat{F}_2 is a distinguishable pair proposed by Dahbura and Masson. Moreover, under the MM* model, the sufficient and necessary condition for two different subsets \widehat{F}_1 and \widehat{F}_2 is a distinguishable pair proposed by Sengupta and Dahbura [11].

Lemma 4 (see [28]). Let $G = (V(G), E(G))$ be a multiprocessor system. For any two distinct sets $\widehat{F}_1, \widehat{F}_2 \subseteq V(G)$, \widehat{F}_1 and \widehat{F}_2 are distinguishable under the PMC model if and only if there exists at least one test from $V(G) - (\widehat{F}_1 \cup \widehat{F}_2)$ to $\widehat{F}_1 \Delta \widehat{F}_2$ (see Figure 1(a)).

Lemma 5 (see [11]). Let $G = (V(G), E(G))$ be a multiprocessor system. For any two distinct sets $\widehat{F}_1, \widehat{F}_2 \subseteq V(G)$, \widehat{F}_1 and \widehat{F}_2 are distinguishable under the MM* model if and only if there is a node $w \in V(G) - \widehat{F}_1 \cup \widehat{F}_2$ such that one of the following conditions holds (see Figure 1(b)):

- (1) $|N_G(w) - (\widehat{F}_1 \cup \widehat{F}_2)| \geq 1$ and $|N_G(w) \cap (\widehat{F}_1 \Delta \widehat{F}_2)| \geq 1$,
- (2) $|N_G(w) \cap (\widehat{F}_1 - \widehat{F}_2)| \geq 2$,
- (3) $|N^{-G}(w) \cap ((F^{-2})^\wedge - (F^{-1})^\wedge)| \geq 2$.

The concept of g -GNCD of a system was proposed in the literature [13].

Definition 6 (see [13]).

- (1) Let $\widehat{F} \subseteq V(G)$ and \widehat{F} be a fault-set. If any node of $V(G) - \widehat{F}$ has at least g neighbors in $G - \widehat{F}$, then \widehat{F} is called a g -good-neighbor conditional fault-set.
- (2) A system G is g -good-neighbor conditional t -diagnosable if each distinct pair of g -good-neighbor conditional faulty (g -GNCF) sets \widehat{F}_1 and \widehat{F}_2 of $V(G)$ with $|\widehat{F}_1| \leq t$ and $|\widehat{F}_2| \leq t$ are distinguishable.

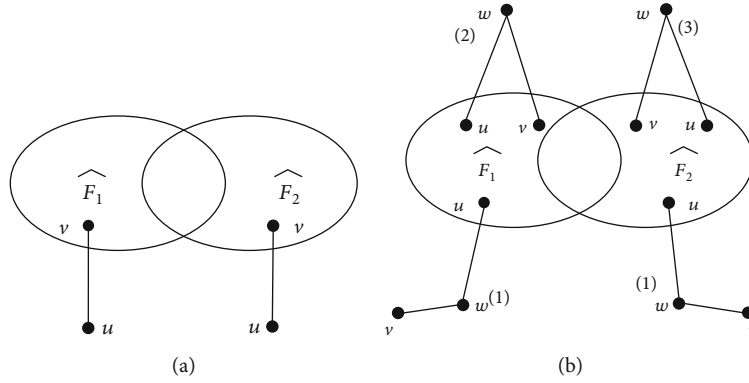


FIGURE 1: (a) An illustration for Lemma 4. (b) An illustration for Lemma 5.

- (3) The g -GNCD, denoted by $t_g(G)$, is defined as the maximum value of t such that G is g -good-neighbor conditionally t -diagnosable. Let $t_g^P(G)$ and $t_g^M(G)$ be the g -GNCD of G under the PMC model and MM^* model, respectively.

2.3. *Generalized Exchanged X-Cubes.* In this subsection, we give the definition of the family of generalized exchanged networks, denoted by *generalized exchanged X-cubes*, which have some common properties, so that their R^g -connectivity (or g -GNCD) can be studied by a uniform method. Since generalized exchanged X-cubes are derived by BC networks (bijective connection networks), we first review the definition of the BC network.

Definition 7 (see [29]). The one-dimensional BC network X_1 contains only two nodes which forms an edge. We use \mathbb{L}_1 to represent the family of the one-dimensional BC network with $\mathbb{L}_1 = \{X_1\}$. A graph G belongs to the family of n -dimensional BC networks \mathbb{L}_n if and only if there exists $V_0, V_1 \subset V(G)$ such that the following two conditions hold:

$$\begin{aligned} V(G) &= V_0 \cup V_1, V_0 \neq \emptyset, V_1 \neq \emptyset, V_0 \cap V_1 \\ &= \emptyset, \text{ and } G[V_0], G[V_1] \in \mathbb{L}_{n-1} \end{aligned} \quad (1)$$

$E(V_0, V_1)$ is a perfect matching M between V_0 and V_1 in G

For any $X_n \in \mathbb{L}_n$, by Definition 7, there exist V_0, V_1, M satisfying the conditions. We use X_{n-1}^0, X_{n-1}^1 to denote the induced subgraph $G[V_0], G[V_1]$, respectively. Clearly, they are both $(n-1)$ -dimensional BC networks, and $E(X_{n-1}^0), E(X_{n-1}^1), M$ is a decomposition of $E(X_n)$. We define the decomposition as $X_n = G(X_{n-1}^0, X_{n-1}^1; M)$.

BC networks are a class of networks containing a number of famous networks such as hypercubes [13], the Möbius cubes [30], crossed cubes [31], and locally twisted cubes [32] as members. An n -dimensional BC network X_n is n -regular and consisting of 2^n nodes. Figure 2 shows two three-dimensional BC networks.

Lemma 8 (see [33]). For $0 \leq g \leq n$ and $Y \subset V(X_n)$, if $\delta(X_n[Y]) \geq g$, then $|Y| \geq 2^g$.

Lemma 9 (see [34]).

- (1) For $Y \subset V(X_n)$ and $0 \leq g \leq n$, if $\delta(X_n[Y]) = g$, then $|N_{X_n}(Y)| \geq (n-g)2^g$.
- (2) For $0 \leq g \leq n-2$, $\kappa^g(X_n) = (n-g)2^g$.

Lemma 10 (see [35]). For $n \geq 2$, there are at most two common neighbors between any two nodes of X_n .

Next, we introduce the definition of generalized exchanged X-cubes.

Definition 11. The (s, t) -dimensional generalized exchanged X-cubes is defined as a graph $GEX(s, t) = (V(GEX(s, t)), E(GEX(s, t)))$, for $s \geq 1$ and $t \geq 1$. $GEX(s, t)$ consists of two disjoint subgraphs \widetilde{L}' and \widetilde{R}' . And \widetilde{L}' consists of 2^t subgraphs, denoted by \widetilde{L}_i' for $i = 1, 2, \dots, 2^t$. Similarly, \widetilde{R}' consists of 2^s subgraphs, denoted by \widetilde{R}_j' for $j = 1, 2, \dots, 2^s$. Moreover, $GE X(s, t)$ satisfies the following conditions (see Figure 3):

- (a) For any integers $1 \leq i \leq 2^t$ and $1 \leq j \leq 2^s$, $\widetilde{L}_i' \cong X_s$ and $\widetilde{R}_j' \cong X_t$. Further, $|V(\widetilde{L}_i')| = 2^s$ and $|V(\widetilde{R}_j')| = 2^t$
- (b) Each node in $V(\widetilde{L}')$ has a sole neighbor in $V(\widetilde{R}')$ and vice versa. In addition, for distinct nodes in each \widetilde{L}_i' , their neighbors of \widetilde{R}' lie in different \widetilde{R}_j'
- (c) For any two different subgraphs \widetilde{L}_i' and \widetilde{L}_h' with $i \neq h$, there exists no edge between them. Similar for \widetilde{R}_j' and \widetilde{R}_k' with $j \neq k$.

By Definition 11, we can deduce that $|V(GEX(s, t))| = 2^{s+t+1}$. Let each of \widetilde{L}_i' and \widetilde{R}_j' be a cluster of $GEX(s, t)$. Obviously, $GEX(s, t)$ consists of $2^t + 2^s$ clusters. If we contract each cluster as a node, then $GEX(s, t)$ is contracted into a complete bipartite graph $K_{2^t, 2^s}$. The edges that connect

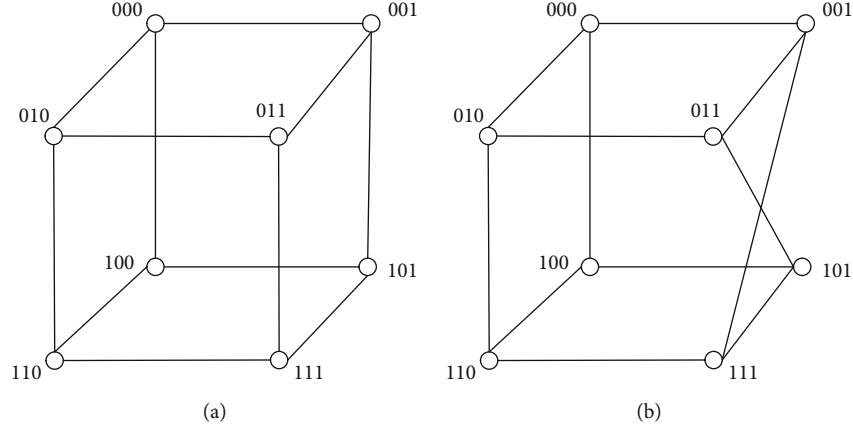
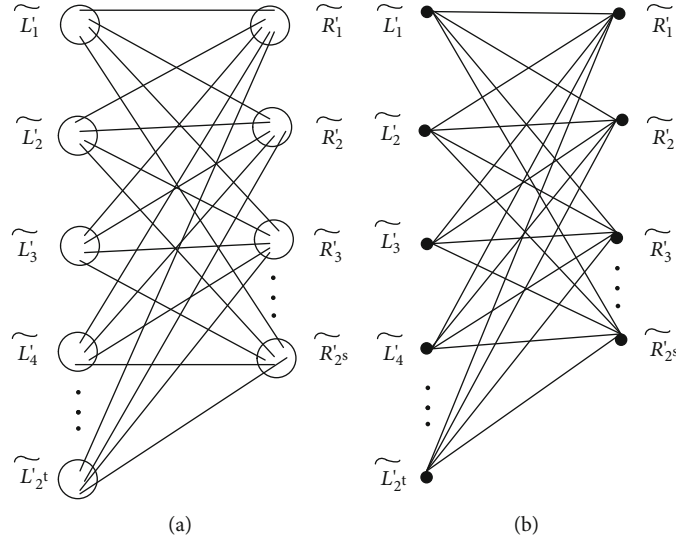


FIGURE 2: Two three-dimensional BC networks.


 FIGURE 3: (a) The partition of $GEX(s, t)$. (b) The contraction of $GEX(s, t)$.

different clusters are called cross edges. In the following discussion, we consider $s \leq t$, and thus, $\delta(GEX(s, t)) = s + 1$, $\Delta(GEX(s, t)) = t + 1$.

3. The R^g -Connectivity of $GEX(s, t)$

In this section, we establish the R^g -connectivity of $GEX(s, t)$ with $1 \leq g \leq s - 2$. In what follows, we exploit some useful lemmas for our further investigation.

Lemma 12. *For any integers $s \geq 3$ and $1 \leq g \leq s$, let H be a subgraph of X_s with $\delta(H) \geq g$, and let T be a subgraph of X_s such that $T \cong X_g$. Then $|N_{X_s}[H]| \geq |N_{X_s}[T]| = (s - g + 1)2^g$.*

Proof. We conduct induction on s .

If $s = 3$, by fixing g , the lemma holds obviously. Suppose that the lemma holds for $s = \tau - 1$, let H be a subgraph of $X_{\tau-1}$ with $\delta(H) \geq g$ and T_1 be a subgraph of $X_{\tau-1}$ such that $T_1 \cong X_g$, then $|N_{X_{\tau-1}}[H]| \geq |N_{X_{\tau-1}}[T_1]| = (\tau - g)2^g$ for $1 \leq g \leq \tau - 1$ and $\tau \geq 4$. In the following, we will prove that

the lemma holds for $s = \tau$. Since X_τ can be merged through a perfect matching by two $X_{\tau-1}$, namely $X_{\tau-1}^0$ and $X_{\tau-1}^1$, we discuss the two cases below.

Case 1. $H \cap X_{\tau-1}^0 \neq \emptyset$ and $H \cap X_{\tau-1}^1 \neq \emptyset$.

Let $H_1 = H - X_{\tau-1}^1$ and $H_2 = H - X_{\tau-1}^0$. Then $\delta(H_1) \geq g - 1$ and $\delta(H_2) \geq g - 1$. Let T_2 and T_3 be two subgraphs of $X_{\tau-1}^0$ and $X_{\tau-1}^1$ with $T_2 \cong X_{g-1}$ and $T_3 \cong X_{g-1}$, respectively. Thus, by the induction hypothesis, we have

$$\begin{aligned} |N_{X_\tau}[H]| &= |N_{X_{\tau-1}}[H_1]| + |N_{X_{\tau-1}}[H_2]| \\ &\geq |N_{X_{\tau-1}}[T_2]| + |N_{X_{\tau-1}}[T_3]| \\ &\geq 2[\tau - g + 1]2^{g-1} = (\tau - g + 1)2^g. \end{aligned} \quad (2)$$

Then, for $s = \tau$, the lemma holds.

Case 2. $H \subseteq X_{\tau-1}^0$ or $H \subseteq X_{\tau-1}^1$.

Without loss of generality, we suppose that $H \subseteq X_{\tau-1}^0$. By Lemma 8, we have $|V(H)| \geq 2^g$. Further, by the induction hypothesis, $|N_{X_{\tau-1}}[H]| \geq |N_{X_{\tau-1}}[T_1]| = (\tau - g)2^g$. Then

$$\begin{aligned} |N_{X_\tau}[H]| &= |N_{X_{\tau-1}}[H]| + |V(H)| \\ &\geq |N_{X_{\tau-1}}[T_1]| + |V(H)| \\ &\geq (\tau - g)2^g + 2^g = (\tau - g + 1)2^g. \end{aligned} \quad (3)$$

Hence, the lemma holds.

Lemma 13. For any integers $s \geq 3$ and $1 \leq g \leq s - 2$, $\kappa^g(GEX(s, t)) \leq (s - g + 1)2^g$.

Proof. By Definition 11, $GEX(s, t)$ can be decomposed into two disjoint subgraphs \widetilde{L}' and \widetilde{R}' , where \widetilde{L}' can be partitioned into 2^t subgraphs (clusters) and \widetilde{R}' can be partitioned into 2^s subgraphs (clusters). Without loss of generality, let $A \subseteq V(\widetilde{L}_1')$ such that $GEX(s, t)[A] \cong X_g$. Clearly, $|A| = 2^g$. By Definition 7 and Lemma 9, $|N_{\widetilde{L}_1'}(A)| = (s - g)2^g$. Further, by Definition 11, each node in $V(\widetilde{L}_1')$ has a sole neighbor in $V(\widetilde{R}')$. And, for distinct nodes in each \widetilde{L}_i' , their neighbors of \widetilde{R}' lie in different \widetilde{R}_j' . In addition, for any two different subgraphs \widetilde{L}_i' and \widetilde{L}_h' with $i \neq h$, the edge between them is nonexistent. Thus, each node in \widetilde{L}_1' has exactly one neighbor in $GEX(s, t) - \widetilde{L}_1'$. Then $|N_{GEX(s, t) - \widetilde{L}_1'}(A)| = |A| = 2^g$. Thus, we have

$$\begin{aligned} |N_{GEX(s, t)}(A)| &= |N_{\widetilde{L}_1'}(A)| + |N_{GEX(s, t) - \widetilde{L}_1'}(A)| \\ &= (s - g)2^g + 2^g = (s - g + 1)2^g \end{aligned} \quad (4)$$

Since $|N_{GEX(s, t)}[A]| = (s - g + 1)2^g + 2^g = (s - g + 2)2^g$ and $|V(GEX(s, t))| = 2^{s+t+1} > (s - g + 2)2^g$, $GEX(s, t) - N_{GEX(s, t)}(A)$ is disconnected. Then $N_{GEX(s, t)}(A)$ is a node-cut of $GEX(s, t)$.

In what follows, $N_{GEX(s, t)}(A)$ as an R^g -cut of $GEX(s, t)$ will be proved. That is, $\delta(GEX(s, t) - N_{GEX(s, t)}(A)) \geq g$.

Since $GEX(s, t)[A] \cong X_g$, $\delta(GEX(s, t)[A]) = g$. By Lemma 9, $N_{\widetilde{L}_1'}(A)$ is an R^g -cut of \widetilde{L}_1' , where $1 \leq g \leq s - 2$. As a result, $\delta(\widetilde{L}_1' - N_{\widetilde{L}_1'}(A)) \geq g$ with $1 \leq g \leq s - 2$. Moreover, by Definition 11, each node in \widetilde{L}_1' has exactly one neighbor in $GEX(s, t) - \widetilde{L}_1'$, and for distinct nodes in \widetilde{L}_1' , their neighbors in \widetilde{R}' lie in different \widetilde{R}_j' . Since $\delta(GEX(s, t)) = s + 1$, $\delta(GEX(s, t) - \widetilde{L}_1' - N_{GEX(s, t) - \widetilde{L}_1'}(A)) \geq s + 1 - 1 > g$ for any node $w \in GEX(s, t) - \widetilde{L}_1' - N_{GEX(s, t) - \widetilde{L}_1'}(A)$.

Summary of the above discussion, we have $\delta(GEX(s, t) - N_{GEX(s, t)}(A)) \geq g$. Then $N_{GEX(s, t)}(A)$ is a g -good-neighbor cut of $GEX(s, t)$. Hence, $\kappa^g(GEX(s, t)) \leq (s - g + 1)2^g$ with $1 \leq g \leq s - 2$ and $s \geq 3$, the lemma holds.

Lemma 16. For any integers $s \geq 3$ and $1 \leq g \leq s - 2$, $\kappa^g(GEX(s, t)) \geq (s - g + 1)2^g$.

Proof. We assume U as a minimum R^g -cut of $GEX(s, t)$. Let $U \cap V(\widetilde{L}_i') = U_{L_i'}$ and $U \cap V(\widetilde{R}_j') = U_{R_j'}$, where $1 \leq i \leq 2^t$, $1 \leq j \leq 2^s$. Then we will show that $\kappa^g(GEX(s, t)) = |U| \geq (s - g + 1)2^g$ with $1 \leq g \leq s - 2$ and $s \geq 3$. We consider three cases as follows.

Case 1. $\widetilde{L}_i' - U_{L_i'}$ and $\widetilde{R}_j' - U_{R_j'}$ are connected for each i, j , where $1 \leq i \leq 2^t$, $1 \leq j \leq 2^s$.

We prove this case by contradiction. Suppose that $|U| \leq (s - g + 1)2^g - 1$. In the following, we will prove that U is not an R^g -cut of $GEX(s, t)$.

Since U is a minimum R^g -cut of $GEX(s, t)$, $GEX(s, t) - U$ is disconnected. In addition, there must exist a component C with C traverses r clusters, where $1 \leq r \leq 2^{s-1}$. Let $U_i = C_i \cap U$, where C_i be one of these r clusters with $1 \leq i \leq r$. As a result, $C = \bigcup_{i=1}^r (C_i - U_i)$. By Definition 11, for any node in $V(\widetilde{L}')$, it has a sole neighbor in $V(\widetilde{R}')$. And, for distinct nodes in each \widetilde{L}_i' , their neighbors of \widetilde{R}' lie in different \widetilde{R}_j' . In addition, for any two different subgraphs \widetilde{L}_i' and \widetilde{L}_h' with $i \neq h$, there exists no edge between them. Then there exist at most $r - 1$ cross edges between $C_i - U_i$ and C_{I-p} , where $I = \{1, 2, \dots, r\}$. Moreover, there are at least $2^s - |U_i| - (r - 1)$ cross edges between $C_i - U_i$ and C_j with $J = \{r + 1, r + 2, \dots, 2^s + 2^t\}$. Clearly, $C_j = \bigcup_{j=r+1}^{2^s+2^t} C_j$ and $U_j = \bigcup_{j=r+1}^{2^s+2^t} U_j$. Since there is no edge between $C_i - U_i$ and $C_j - U_j$, $|U_j| \geq \sum_{i=1}^r [2^s - |U_i| - (r - 1)]$. Then, we have

$$\begin{aligned} |U| &= |U_1| + |U_2| + \dots + |U_r| + |U_J| \\ &\geq |U_1| + |U_2| + \dots + |U_r| + \sum_{i=1}^r [2^s - |U_i| - (r - 1)] \\ &= r[2^s - r + 1]. \end{aligned} \quad (5)$$

Let $f(r) = r(2^s - r + 1)$ with $1 \leq r \leq 2^{s-1}$. We obtain $\partial f(r)/\partial r = 2^s - 2r + 1 > 0$. Thus, $f(r)$ is an increasing function. Therefore, $f(r) \geq f(1) = 2^s$ and $|U| \geq f(r) \geq 2^s$. In addition, let $f(g) = 2^s - [(s - g + 1)2^g - 1]$ with $1 \leq g \leq s - 2$. We obtain that $\partial f(g)/\partial g = 2^g [g \ln 2 + 1 - (s + 1) \ln 2] < 0$. Thus, $f(g)$ is a decreasing function. Therefore, $f(g) \geq f(s - 2) = 2^{s-2} + 1 > 0$. Then $|U| \geq 2^s > (s - g + 1)2^g - 1$, which results in a contradiction with $|U| \leq (s - g + 1)2^g - 1$.

Case 2. Only one of $\widetilde{L}_i' - U_{L_i'}$ and $\widetilde{R}_j' - U_{R_j'}$ is disconnected, where $1 \leq i \leq 2^t$, $1 \leq j \leq 2^s$.

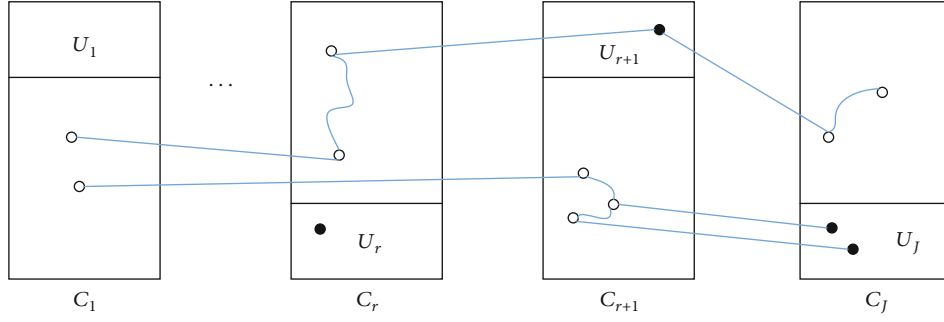


FIGURE 4: An illustration for Case 2 of Lemma 16.

Without loss of generality, assume that $\widetilde{L}_1' - U_{L_1}'$ is disconnected. Since U is an R^g -cut of $GEX(s, t)$, $\delta(\widetilde{L}_1' - U_{L_1}') \geq g - 1$. Then U_{L_1}' is a $(g - 1)$ -good-neighbor cut of $GEX(s, t)$. By Lemma 9, $|U_{L_1}'| \geq (s - g + 1)2^{g-1}$. By contradiction, suppose that $|U| \leq (s - g + 1)2^g - 1$ with $1 \leq g \leq s - 2$ and $s \geq 3$. Let $U_{L_1}' = U_1$ and $U_I = \bigcup_{i=2}^{2^s+2^t} U_i$. Then $|U_I| = |U - U_1| \leq (s - g + 1)2^{g-1} - 1$.

Assume that $GEX(s, t) - \widetilde{L}_1' - U_I$ is disconnected. Then there must exist a component C such that C traverses r clusters, where $1 \leq r \leq \lfloor 2^{s-1} - 1 \rfloor$. Let C_i be one of these r clusters for $2 \leq i \leq r + 1$ and $U_i = C_i \cap U$. As a result, $C = \bigcup_{i=2}^{r+1} (C_i - U_i)$. By Definition 11, for any node in $V(\widetilde{L}_1')$, it has a sole neighbor in $V(\widetilde{R}_1')$. And for distinct nodes in each \widetilde{L}_i' , their neighbors of \widetilde{R}_1' lie in different \widetilde{R}_j' . In addition, for any two different subgraphs \widetilde{L}_i' and \widetilde{L}_h' with $i \neq h$, the edge between them is nonexistent. Then there exist at most r cross edges between $C_i - U_i$ and $C_{I'} - U_{I'}$, where $I' = \{1, 2, \dots, r, r + 1\}$. Moreover, there are at least $2^s - |U_i| - r$ cross edges between $C_i - U_i$ and C_J with $J = \{r + 2, r + 3, \dots, 2^s + 2^t\}$. Clearly, $C_J = \bigcup_{j=r+2}^{2^s+2^t} C_j$ and $U_J = \bigcup_{j=r+2}^{2^s+2^t} U_j$. Since there is no edge between $C_i - U_i$ and $C_J - U_J$, $|U_J| \geq \sum_{i=2}^{r+1} [2^s - |U_i| - r]$. Figure 4 shows an illustration for this case. Then, we have

$$\begin{aligned} |U_I| &= |U_2| + \dots + |U_{r+1}| + |U_J| \\ &\geq |U_2| + \dots + |U_{r+1}| + \sum_{i=2}^{r+1} [2^s - |U_i| - r] = r(2^s - r). \end{aligned} \quad (6)$$

Let $f(r) = r(2^s - r)$ with $1 \leq r \leq \lfloor 2^{s-1} - 1 \rfloor$. We obtain that $\partial f(r)/\partial r = 2^s - 2r > 0$. Thus, $f(r)$ is an increasing function. Therefore, $f(r) \geq f(1) = 2^s - 1$. And $|U_I| \geq f(r) \geq 2^s - 1$. In addition, let $f(g) = 2^s - 1 - [(s - g + 1)2^{g-1} - 1]$ with $1 \leq g \leq s - 2$ and $s \geq 3$. We get that $\partial f(g)/\partial g = 2^{g-1}[g \ln 2 + 1 - (s + 1) \ln 2] < 0$. Thus, $f(g)$ is a decreasing function. Therefore, $f(g) \geq f(s - 2) > 0$. Then $|U_I| \geq 2^s - 1 > (s - g + 1)2^{g-1} - 1$, which results in a contradiction with $|U_I| \leq (s - g + 1)2^{g-1} - 1$.

Thus, $GEX(s, t) - \widetilde{L}_1' - U_I$ is connected. Since $\widetilde{L}_1' - U_1$ is disconnected, there must exist a component H in $\widetilde{L}_1' - U_1$ such that there is no edge between H and $GEX(s, t) - \widetilde{L}_1' - U_I$. Then $|U_I| \geq |V(H)|$. Since $N_{L_1}'(H) \subseteq U_1$, $|U| = |U_1| + |U_I| \geq |N_{L_1}'(H)| + |V(H)| = |N_{L_1}'[H]|$. Moreover, since U is an R^g -cut of $GEX(s, t)$, $\delta(H) \geq g$. By Lemma 8 and $\widetilde{L}_1' \cong X_s$, we have $|U| = |N_{L_1}'[H]| \geq (s - g + 1)2^g$, which results in a contradiction with $|U| \leq (s - g + 1)2^g - 1$.

Case 3. For any integers $1 \leq i \leq 2^t$, $1 \leq j \leq 2^s$, there are at least two of $\widetilde{L}_i' - U_{L_i}'$ and $\widetilde{R}_j' - U_{R_j}'$ that are disconnected.

Without loss of generality, suppose that $\widetilde{L}_1' - U_{L_1}'$ and $\widetilde{L}_2' - U_{L_2}'$ are disconnected. Since U is an R^g -cut of $GEX(s, t)$ and by Definition 11, we have $\delta(\widetilde{L}_1' - U_{L_1}') \geq g - 1$ and $\delta(\widetilde{L}_2' - U_{L_2}') \geq g - 1$. Then U_{L_1}' and U_{L_2}' are two $(g - 1)$ -good-neighbor cuts of $GEX(s, t)$. By Lemma 9, $|U_{L_1}'| \geq (s - g + 1)2^{g-1}$ and $|U_{L_2}'| \geq (s - g + 1)2^{g-1}$. Then

$$\begin{aligned} |U| &\geq |U_{L_1}'| + |U_{L_2}'| \geq (s - g + 1)2^{g-1} + (s - g + 1)2^{g-1} \\ &= (s - g + 1)2^g. \end{aligned} \quad (7)$$

Thus, $\kappa^g(GEX(s, t)) = |U| \geq (s - g + 1)2^g$.

Hence, the lemma holds.

Combining Lemma 13 and Lemma 16, the following theorem holds.

Theorem 14. For any integers $s \geq 3$ and $1 \leq g \leq s - 2$, $\kappa^g(GEX(s, t)) = (s - g + 1)2^g$.

4. The g -Good-Neighbor Conditional Diagnosability of $GEX(s, t)$

In this section, we will determine the g -GNCD of $GEX(s, t)$ under the PMC model and MM* model, respectively, where $1 \leq g \leq s - 2$.

Theorem 15. For any integers $s \geq 3$ and $1 \leq g \leq s-2$, $t_g^P(GEX(s, t)) = (s-g+2)2^g - 1$.

Proof. First, we show that $t_g^P(GEX(s, t)) \leq (s-g+2)2^g - 1$ with $1 \leq g \leq s-2$ and $s \geq 3$. Let $A \subseteq V(C_1)$ with $C_1 \cong X_s$ such that $GEX(s, t)[A] \cong X_g$. Clearly, $|A| = 2^g$. Suppose that $\widehat{F}_1 = N_{GEX(s, t)}(A)$ and $\widehat{F}_2 = N_{GEX(s, t)}[A]$. By Lemma 13, we have $|\widehat{F}_1| = |N_{GEX(s, t)}(A)| = (s-g+1)2^g$, and $|\widehat{F}_2| = |N_{GEX(s, t)}[A]| = (s-g+2)2^g$, where $\delta(GEX(s, t) - \widehat{F}_2) \geq g$. Since $\delta(GEX(s, t) - \widehat{F}_1) \geq g$ and $\delta(GEX(s, t) - \widehat{F}_2) \geq g$, \widehat{F}_1 and \widehat{F}_2 are two g -GNCF sets of $GEX(s, t)$ with $|\widehat{F}_1| \leq (s-g+2)2^g$ and $|\widehat{F}_2| \leq (s-g+2)2^g$. On the other hand, since $V(A) = \widehat{F}_1 \Delta \widehat{F}_2$ and $N_{GEX(s, t)}(A) = \widehat{F}_1$, there is no edge between $\widehat{F}_1 \Delta \widehat{F}_2$ and $GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2$. By Lemma 4, \widehat{F}_1 and \widehat{F}_2 are indistinguishable under the PMC model. By Definition 6 (2), $GEX(s, t)$ is not g -good-neighbor conditional $(s-g+2)2^g$ -diagnosable under the PMC model. That is, $t_g^P(GEX(s, t)) \leq (s-g+2)2^g - 1$ for $1 \leq g \leq s-2$.

Next, we prove that $t_g^P(GEX(s, t)) \geq (s-g+2)2^g - 1$ with $1 \leq g \leq s-2$ and $s \geq 3$. We suppose, to the contrary, that $t_g^P(GEX(s, t)) \leq (s-g+2)2^g - 2$ for $1 \leq g \leq s-2$. And assume that there are two indistinguishable g -GNCF sets \widehat{F}_1 and \widehat{F}_2 with $|\widehat{F}_1| \leq (s-g+2)2^g - 1$ and $|\widehat{F}_2| \leq (s-g+2)2^g - 1$. In what follows, we consider two cases.

Case 1. $V(GEX(s, t)) = \widehat{F}_1 \cup \widehat{F}_2$.

Since $s \leq t$, by Definition 11, we have $|V(GEX(s, t))| = 2^{s+t+1} \geq 2^{2s+1}$.

Since $|\widehat{F}_1 \cup \widehat{F}_2| \leq |\widehat{F}_1| + |\widehat{F}_2| \leq 2[(s-g+2)2^g - 1]$, we have

$$|V(GEX(s, t))| - |\widehat{F}_1 \cup \widehat{F}_2| \geq 2^{2s+1} - 2[(s-g+2)2^g - 1]. \quad (8)$$

Let $f(g) = 2^{2s+1} - (s-g+2)2^{g+1} + 2$ with $1 \leq g \leq s-2$ and $s \geq 3$. We obtain that $\partial f(g)/\partial g = [(g-s-2) \ln 2 + 1] 2^{g+1} < 0$. Thus, $f(g)$ is a decreasing function. Therefore, for $1 \leq g \leq s-2$ and $s \geq 3$,

$$f(g) \geq f(s-2) = 2^{2s+1} - 2^{s+1} + 2 > 0, \quad (9)$$

which induces a contradiction since $V(GEX(s, t)) = \widehat{F}_1 \cup \widehat{F}_2$.

Case 2. $V(GEX(s, t)) \neq \widehat{F}_1 \cup \widehat{F}_2$.

Since $\widehat{F}_1 \neq \widehat{F}_2$, we may assume that $\widehat{F}_2 - \widehat{F}_1 \neq \emptyset$. There exists no edge between $V(GEX(s, t)) - \widehat{F}_1 \cup \widehat{F}_2$ and $\widehat{F}_1 \Delta \widehat{F}_2$ because \widehat{F}_1 and \widehat{F}_2 are indistinguishable. Moreover, since \widehat{F}_1 is a g -good-neighbor conditional faulty set, it is easy to verify that $\delta(GEX(s, t)[\widehat{F}_2 - \widehat{F}_1]) \geq g$. By Lemma 8, $|\widehat{F}_2 - \widehat{F}_1| \geq 2^g$. On the other hand, since both \widehat{F}_1 and \widehat{F}_2 are g -

GNCF sets, $\widehat{F}_1 \cap \widehat{F}_2$ is also a g -good-neighbor conditional faulty set. Moreover, there is no edge between $V(GEX(s, t)) - \widehat{F}_1 \cup \widehat{F}_2$ and $\widehat{F}_1 \Delta \widehat{F}_2$; thus, $GEX(s, t) - \widehat{F}_1 \cap \widehat{F}_2$ is disconnected. Then $\widehat{F}_1 \cap \widehat{F}_2$ is an R^g -cut of $GEX(s, t)$. By Theorem 14, $|\widehat{F}_1 \cap \widehat{F}_2| \geq (s-g+1)2^g$ with $s \geq 3$ and $1 \leq g \leq s-2$. Hence,

$$\begin{aligned} |\widehat{F}_2| &= |\widehat{F}_2 - \widehat{F}_1| + |\widehat{F}_1 \cap \widehat{F}_2| \geq 2^g + (s-g+1)2^g \\ &= (s-g+2)2^g, \end{aligned} \quad (10)$$

which results in a contradiction since $|\widehat{F}_2| \leq (s-g+2)2^g - 1$.

To sum up, we can conclude that $t_g^P(GEX(s, t)) \geq (s-g+2)2^g - 1$ for any integers $1 \leq g \leq s-2$ and $s \geq 3$.

Hence, the theorem holds.

Theorem 24. For any integers $1 \leq g \leq s-2$ and $s \geq 4$, $t_g^M(GEX(s, t)) = (s-g+2)2^g - 1$.

Proof. The proof of $t_g^M(GEX(s, t)) \leq (s-g+2)2^g - 1$ with $1 \leq g \leq s-2$ and $s \geq 4$ is similar to Theorem 15, so it is omitted.

Next, we prove that $t_g^M(GEX(s, t)) \geq (s-g+2)2^g - 1$ with $s \geq 4$ and $1 \leq g \leq s-2$. We suppose, to the contrary, that $t_g^M(GEX(s, t)) \leq (s-g+2)2^g - 2$ with $s \geq 4$ and $1 \leq g \leq s-2$. Moreover, we assume that there are two indistinguishable g -GNCF sets \widehat{F}_1 and \widehat{F}_2 , where $|\widehat{F}_1| \leq (s-g+2)2^g - 1$ and $|\widehat{F}_2| \leq (s-g+2)2^g - 1$.

Since $s \leq t$, by Definition 11, we have $|V(GEX(s, t))| = 2^{s+t+1} \geq 2^{2s+1}$.

Furthermore, it is easy to get that $|\widehat{F}_1 \cup \widehat{F}_2| \leq |\widehat{F}_1| + |\widehat{F}_2| \leq 2[(s-g+2)2^g - 1]$. Then, we have

$$|V(GEX(s, t))| - |\widehat{F}_1 \cup \widehat{F}_2| \geq 2^{2s+1} - 2[(s-g+2)2^g - 1]. \quad (11)$$

Let $f(g) = 2^{2s+1} - (s-g+2)2^{g+1} + 2$ with $s \geq 4$ and $1 \leq g \leq s-2$. We obtain that $\partial f(g)/\partial g = [g \ln 2 + 1 - (s+2) \ln 2] 2^{g+1} < 0$. Thus, $f(g)$ is a decreasing function. Therefore, for $s \geq 4$ and $1 \leq g \leq s-2$,

$$f(g) \geq f(s-2) = 2^{2s+1} - 2^{s+1} + 2 > 0, \quad (12)$$

which results in a contradiction since $|\widehat{F}_2| \leq (s-g+2)2^g - 1$.

Thus, $V(GEX(s, t)) \neq \widehat{F}_1 \cup \widehat{F}_2$. In addition, an important claim is given as follows.

Claim 25. $GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2$ has no isolated node.

By contradiction, suppose that $GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2$ has at least one isolated node. Then, we prove that the two cases both contradict the supposition.

Case 1. $g = 1$.

Since $\widehat{F}_1 \neq \widehat{F}_2$, without loss of generality, we suppose that $\widehat{F}_2 - \widehat{F}_1 \neq \emptyset$. When $\widehat{F}_1 \subset \widehat{F}_2$, since \widehat{F}_2 is a 1-GNCF set, $GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2$ has no isolated node. Now, we consider $\widehat{F}_1 \not\subset \widehat{F}_2$. The given W is the set of all isolated nodes and $B = GEX(s, t)[V(GEX(s, t)) - (\widehat{F}_1 \cup \widehat{F}_2) - W]$. Since \widehat{F}_1 is a 1-GNCF set, $|N_{GEX(s, t) - \widehat{F}_1}^{\widehat{F}_1}(w)| \geq 1$ for any $w \in W$.

Since \widehat{F}_1 and \widehat{F}_2 are indistinguishable, there exists at most one node $u \in \widehat{F}_2 - \widehat{F}_1$ with u is adjacent to w by Lemma 5. Thereby, there exists only one node $u \in \widehat{F}_2 - \widehat{F}_1$ with u adjacent to w . It is easy to see that there is only one node $v \in \widehat{F}_1 - \widehat{F}_2$ with v adjacent to w . Since $\Delta(GEX(s, t)) = t + 1$, there are at most $t - 1$ neighbors of w in $\widehat{F}_1 \cap \widehat{F}_2$ with any isolated node $w \in W$. Since $|\widehat{F}_2| \leq (s - g + 2)2^g - 1$ and $g = 1$, $|\widehat{F}_2| \leq 2s + 1$. Hence,

$$\begin{aligned} & \sum_{w \in W} \left| N_{GEX(s, t)}[\widehat{F}_1 \cap \widehat{F}_2](w) \right| \\ & \leq |W|(t - 1) \leq \sum_{x \in \widehat{F}_1 \cap \widehat{F}_2} \deg_{GEX(s, t)}(x) \\ & \leq |\widehat{F}_1 \cap \widehat{F}_2|(t + 1) \\ & \leq \left(|\widehat{F}_2| - 1 \right)(t + 1) \leq 2s(t + 1). \end{aligned} \quad (13)$$

It follows that $|W| \leq 2s(t + 1)/t - 1 \leq 4s$. Thus,

$$\begin{aligned} |\widehat{F}_1 \cup \widehat{F}_2| + |W| &= |\widehat{F}_1| + |\widehat{F}_2| - |\widehat{F}_1 \cap \widehat{F}_2| + |W| \\ &\leq 2(2s + 1) - (s - 1) + |W| \leq 7s + 3. \end{aligned} \quad (14)$$

Let $f(s) = 2^{2s} - 7s - 3$. We can deduce that $\partial f(s)/\partial s > 0$. Then $f(s)$ is an increasing function. Therefore, $f(s) \geq f(4) > 0$, a contradiction. Thus, $V(B) \neq \emptyset$.

Since the fault-pair $(\widehat{F}_1, \widehat{F}_2)$ does not satisfy Lemma 5 and any node in $V(B)$ is not isolated, there exists no edge between $V(B)$ and $\widehat{F}_1 \Delta \widehat{F}_2$. Moreover, $\widehat{F}_1 \cap \widehat{F}_2$ is also a 1-GNCF set. Thus, $\widehat{F}_1 \cap \widehat{F}_2$ is an R^1 -cut of $GEX(s, t)$. By Theorem 14, $|\widehat{F}_1 \cap \widehat{F}_2| \geq 2s$. Since $|\widehat{F}_1| \leq 2s + 1$, $|\widehat{F}_2| \leq 2s + 1$, and $\widehat{F}_1 \neq \widehat{F}_2$, $|\widehat{F}_1 - \widehat{F}_2| = |\widehat{F}_2 - \widehat{F}_1| = 1$. Let $\widehat{F}_1 - \widehat{F}_2 = \{v\}$ and $\widehat{F}_2 - \widehat{F}_1 = \{u\}$. Then $|N_{GEX(s, t)[\widehat{F}_1 - \widehat{F}_2]}^{\widehat{F}_1}(w)| = |N_{GEX(s, t)[\widehat{F}_2 - \widehat{F}_1]}^{\widehat{F}_2}(w)| = 1$. Hence, we have $(v, w) \in E(GEX(s, t))$ and $(u, w) \in E(GEX(s, t))$ for any isolated node $w \in W$. By Lemma 10, there are at most two common neighbors between any two nodes in $V(X_s)$. In addition, by Definition 11, any two cross edges have no common end node. Then we deduce that any two nodes in $V(GEX(s, t))$ have at most two common neighbors. Thus, $|W| \leq 2$. Since there is no common node between any two cross edges and X_s is triangle-free, $GEX(s, t)$ is triangle-free. Thereby,

$$\begin{aligned} \left| \widehat{F}_1 \cap \widehat{F}_2 \right| &\geq \left| N_{GEX(s, t)}(v) - W \right| + \left| N_{GEX(s, t)}(w) - \{u, v\} \right| \\ &\quad + \left| N_{GEX(s, t)}(v) - W \right| - (2 - |W|) \\ &\geq 2(s + 1 - |W|) + s - 1 - 2 + |W| \\ &= 3s - 1 - |W| \geq 3s - 3. \end{aligned} \quad (15)$$

Therefore, for $s \geq 4$, we have

$$\left| \widehat{F}_2 \right| = \left| \widehat{F}_2 - \widehat{F}_1 \right| + \left| \widehat{F}_1 \cap \widehat{F}_2 \right| \geq 1 + 3s - 3 = 3s - 2 > 2s + 1, \quad (16)$$

which results in a contradiction since $|\widehat{F}_2| \leq 2s + 1$.

Case 2. $g \geq 2$.

Without loss of generality, we suppose that $\widehat{F}_2 - \widehat{F}_1 \neq \emptyset$. Since \widehat{F}_1 is a g -GNCF set of $GEX(s, t)$, $|N_{GEX(s, t) - \widehat{F}_1}^{\widehat{F}_1}(x)| \geq g$ with any node $x \in V(GEX(s, t) - \widehat{F}_1)$. For any $w \in V(GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2)$, there exists at most one neighbor in $\widehat{F}_2 - \widehat{F}_1$ because the fault-pair $(\widehat{F}_1, \widehat{F}_2)$ is indistinguishable by Lemma 5. Therefore, $|N_{GEX(s, t) - (\widehat{F}_1 \cup \widehat{F}_2)}^{\widehat{F}_1}(w)| \geq g - 1 \geq 1$.

Since $w \in V(GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2)$ is arbitrary, every node of $GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2$ is not an isolated one.

To sum up, Claim 25 holds.

Since there exists no isolated node in $GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2$ by Claim 25 we have, for any $w \in GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2$, there exists some node $v \in GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2$ such that $(w, v) \in E(GEX(s, t))$. If $(u, w) \in E(GEX(s, t))$ for any $u \in \widehat{F}_1 \Delta \widehat{F}_2$, $(u, v)_w$ satisfies condition in Lemma 5. Therefore, the g -GNCF sets \widehat{F}_1 and \widehat{F}_2 are distinguishable, which results in a contradiction. By the arbitrariness of $w \in GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2$, there exists no edge between $V(GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2)$ and $\widehat{F}_1 \Delta \widehat{F}_2$.

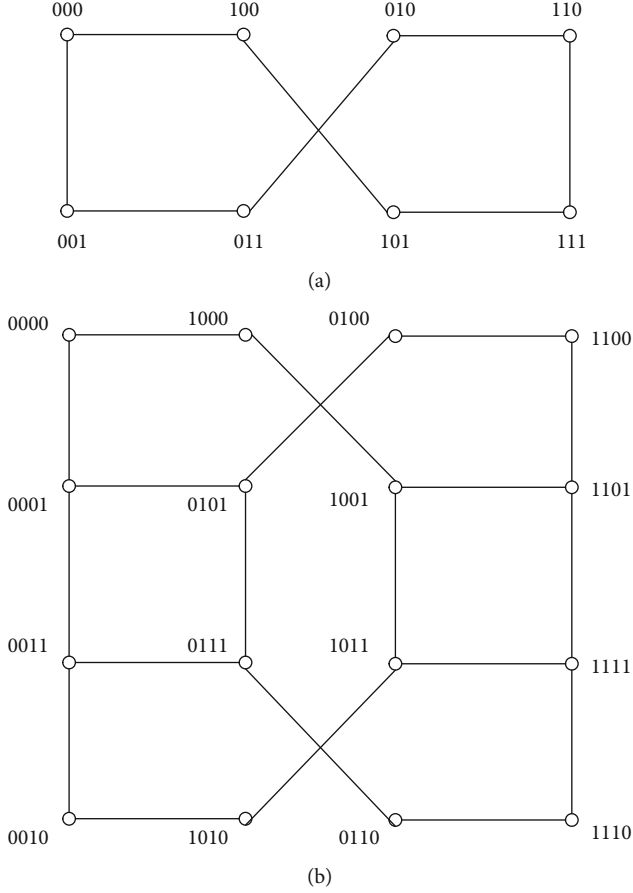
Since \widehat{F}_1 is a g -GNCF set and $\widehat{F}_2 - \widehat{F}_1 \neq \emptyset$, $\delta(GEX(s, t)[\widehat{F}_2 - \widehat{F}_1]) \geq g$. Thus, $|\widehat{F}_2 - \widehat{F}_1| \geq g + 1$. Since \widehat{F}_1 and \widehat{F}_2 are both g -GNCF sets and there exists no edge between $V(GEX(s, t) - \widehat{F}_1 \cup \widehat{F}_2)$ and $\widehat{F}_1 \Delta \widehat{F}_2$, $\widehat{F}_1 \cap \widehat{F}_2$ is also an R^g -cut of $GEX(s, t)$. By Theorem 14, $|\widehat{F}_1 \cap \widehat{F}_2| \geq (s - g + 1)2^g$. Then $|\widehat{F}_2| = |\widehat{F}_2 - \widehat{F}_1| + |\widehat{F}_1 \cap \widehat{F}_2| \geq (s - g + 1)2^g + 2^g = (s - g + 2)2^g$, which contradicts with $|\widehat{F}_2| \leq (s - g + 2)2^g - 1$.

Thus, $t_g^M(GEX(s, t)) \geq (s - g + 2)2^g - 1$ for any integers $1 \leq g \leq s - 2$ and $s \geq 4$.

Hence, the proof of theorem is completed.

5. Applications to a Family of Famous Networks

In Section 2, the definition of the generalized exchanged X -cube $GEX(s, t)$ has been given. Furthermore, we determine the R^g -connectivity and g -GNCD of $GEX(s, t)$ in Section 3 and Section 4, respectively. Applying the theorems of

FIGURE 5: (a) $GEH(1, 1)$, (b) $GEH(1, 2)$.

Section 3 and Section 4, we can directly establish the R^g -connectivity and g -GNCD of some generalized exchanged X -cubes, including generalized exchanged hypercubes, dual-cube-like networks, generalized exchanged crossed cubes, and locally generalized exchanged twisted cubes. In this section, we will give the applications to these networks.

5.1. The Generalized Exchanged Hypercube. In 2005, Loh et al. [36] proposed the exchanged hypercube, which obtained by removing edges from a hypercube H_{s+t+1} . We denote $I_r = \{1, 2, \dots, r\}$, where r is a given position integer. For each $n \in I_r$, the sequence $x_r, x_{r-1} \dots x_1$ is a binary string of length r if $x_n \in \{0, 1\}$. The definition of exchanged hypercubes is presented as follows.

Definition 16 (see [36]). Let $s, t \geq 1$, the exchanged hypercube $EH(s, t)$ consists of the node set $V(EH(s, t))$ and the edge set $E(EH(s, t))$, two nodes $u = u_{s+t} \dots u_{t+1} u_t \dots u_1 u_0$ and $v = v_{s+t} \dots v_{t+1} v_t \dots v_1 v_0$ are linked by an edge, called r -dimensional edge, if and only if the following conditions are satisfied:

- (i) u and v differ exactly in one bit on the r -th bit or on the last bit
- (ii) if $r \in I_t$, then $u_0 = v_0 = 1$

- (iii) $r \in I_{s+t} - I_t$, then $u_0 = v_0 = 0$.

The generalized exchange hypercube was proposed by Cheng et al. [37]. Let $s, t \geq 1$, the generalized exchanged hypercube $GEH(s, t, f)$ consists of two classes of hypercubes: one class contains $2^t H_s$'s, referred to as the Class-0 clusters; and the other contains $2^s H_t$'s, referred to as the Class-1 clusters. Class-0 and Class-1 clusters will be referred to as clusters of opposite class of each other, same class otherwise. The function f is a bijection between nodes of Class-0 clusters and those of Class-1 clusters; for two nodes u, v in the same cluster, $f(u)$ and $f(v)$ are in two different clusters, and the edge $(u, f(u))$ is a cross edge. The bijection f ensures the existence of a perfect matching between nodes of Class-0 clusters and those in the Class-1 clusters but ignores the specifics of the perfect matching. Hence, we present the following proposition.

Proposition 17. $GEH(s, t)$ can be decomposed into two subgraphs \tilde{L}' and \tilde{R}' . Further, \tilde{L}' can be partitioned into 2^t subgraphs, denoted by \tilde{L}_i' for $i = 1, 2, \dots, 2^t$. Similarly, \tilde{R}' can be partitioned into 2^s subgraphs, denoted by \tilde{R}_j' for $j = 1, 2, \dots, 2^s$. And $GEH(s, t)$ satisfies the following conditions (Figure 5 shows the $GEH(1, 1)$ and $GEH(1, 2)$):

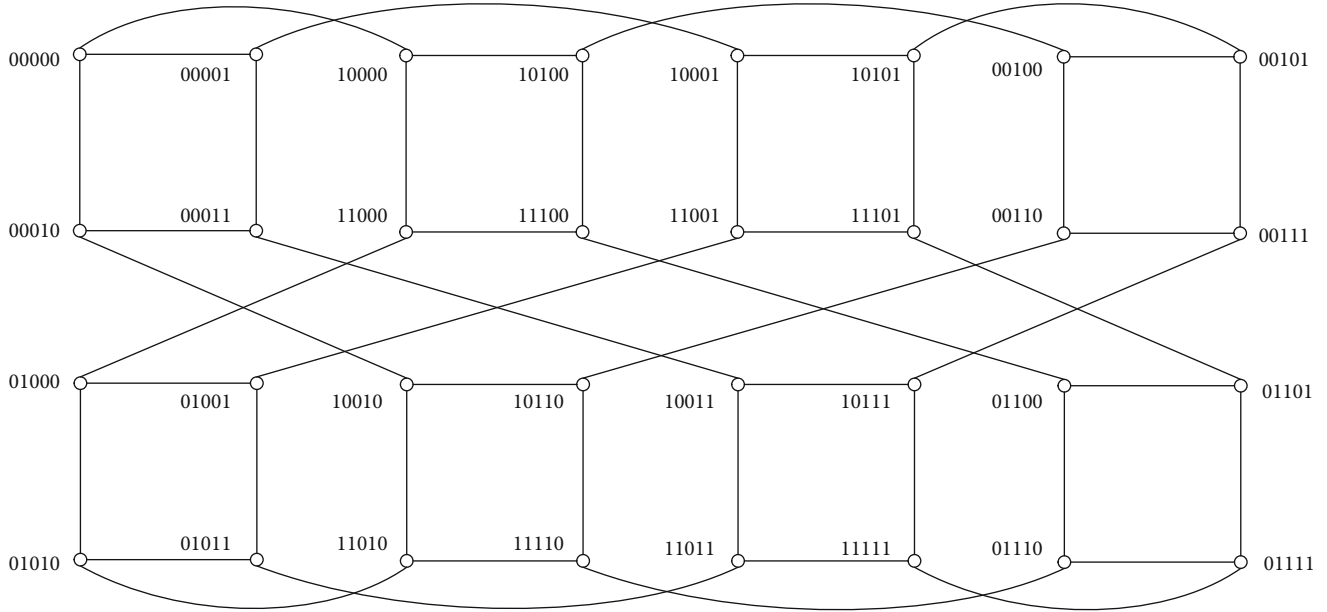
- (a) For any i, j , $\tilde{L}_i' \cong H_s$ and $\tilde{R}_j' \cong H_t$. Further, $|V(\tilde{L}_i')| = 2^s$ and $|V(\tilde{R}_j')| = 2^t$
- (b) Each node in $V(\tilde{L}')$ has a sole neighbor in $V(\tilde{R}')$ and vice versa. In addition, for distinct nodes in each \tilde{L}_i' , their neighbors of \tilde{R}' lie in different \tilde{R}_j'
- (c) For any two different subgraphs \tilde{L}_i' and \tilde{L}_h' with $i \neq h$, there exists no edge between them. Similar for \tilde{R}_j' and \tilde{R}_k' with $j \neq k$.

The dual-cube is a special case of the exchanged hypercube when $s = t$, proposed by Li and Peng [38]. That is, $EH(n, n) \cong D_n$. The dual-cube-like network DC_n [39], which is a generalization of dual-cubes, is isomorphic to $EH(n-1, n-1)$, a special case of $GEH(n-1, n-1)$ (see DC_3 in Figure 6).

By Proposition 17, the generalized exchanged hypercube $GEH(s, t)$ is the member of generalized exchanged X -cubes, where the X -cube is a hypercube. Then, the following theorems hold obviously.

Theorem 18.

- (1) For any integers $1 \leq g \leq s-2$ and $3 \leq s \leq t$, $\kappa^g(GE H(s, t)) = (s-g+1)2^g$
- (2) For any integers $1 \leq g \leq n-3$ and $n \geq 4$, $\kappa^g(DC_n) = (n-g)2^g$.

FIGURE 6: The dual-cube-like network DC_3 .**Theorem 19.**

- (1) For any integers $1 \leq g \leq s-2$ and $3 \leq s \leq t$, $t_g^P(\text{GEH}(s, t)) = (s-g+2)2^g - 1$
- (2) For any integers $1 \leq g \leq n-3$ and $n \geq 4$, $t_g^P(\text{DC}_n) = (n-g+1)2^g - 1$.

Theorem 20.

- (1) For any integers $1 \leq g \leq s-2$ and $4 \leq s \leq t$, $t_g^M(\text{GEH}(s, t)) = (s-g+2)2^g - 1$
- (2) For any integers $1 \leq g \leq n-3$ and $n \geq 5$, $t_g^M(\text{DC}_n) = (n-g+1)2^g - 1$.

5.2. *The Generalized Exchanged Crossed Cube.* Li et al. [26] give the definition of $ECQ(s, t)$, which is obtained by removing edges from a crossed cube CQ_{s+t+1} . In what follows, we review the definition of exchanged crossed cubes.

Definition 21 (see [26]). The (s, t) -dimensional exchanged crossed cube is defined as a graph $ECQ(s, t) = G(V(ECQ(s, t)), E(ECQ(s, t)))$ for $s, t \geq 1$. The node set $V(ECQ(s, t)) = \{a_{s-1} \cdots a_0 b_{t-1} \cdots b_0 c | a_j, b_j, c \in \{0, 1\}, \text{ where } 0 \leq i \leq t-1 \text{ and } 0 \leq j \leq s-1\}$. The edge set $E(ECQ(s, t))$ consisting of three types of disjoint sets E_1, E_2 , and E_3 is shown as follows.

$E_1 : |u[0] \neq v[0], u \oplus v = 1$, where \oplus is the exclusive-OR operator.

$E_2 : |u[s+t:t+1] = v[s+t:t+1], u[0] = v[0] = 1, u[t:1]$ is denoted by $b = b_{t-1} \cdots b_0$ and $v[t:1]$ is denoted by $b' = b'_{t-1} \cdots b'_0$. And u and v are adjacent by the following rule: for any integer $t \geq 1$, if and only if there is an $l(1 \leq l \leq t)$ with

$b_{t-1} \cdots b_l = b'_{t-1} \cdots b'_l; b_{l-1} \neq b'_{l-1}, b_{l-2} = b'_{l-2}$ if l is even; $b_{2i+1}b_{2i} \sim b'_{2i+1}b'_{2i}$, where $0 \leq i < \lfloor (l-1)/2 \rfloor$
 $E_3 : |u[t:1] = v[t:1], u[0] = v[0] = 0, u[s+t:t+1]$ is denoted by $a = a_{s-1} \cdots a_0$ and $y[s+t:t+1]$ is denoted by $a' = a'_{t-1} \cdots a'_0$. And u and v are adjacent by the following rule: for any integer $s \geq 1$, if and only if there is an $l(1 \leq l \leq s)$ with $a_{s-1} \cdots a_l = a'_{s-1} \cdots a'_l; a_{l-1} \neq a'_{l-1}, a_{l-2} = a'_{l-2}$ if l is even; $a_{2i+1}a_{2i} \sim a'_{2i+1}a'_{2i}$, where $0 \leq i < \lfloor (l-1)/2 \rfloor$.
 $x[u:v]$ is the bit pattern of x from dimension u to dimension v .

Let $s, t \geq 1$, the generalized crossed cube $GECQ(s, t, f)$ comprises two classes of crossed cubes, referred to as the Class-0 clusters and the Class-1 clusters, respectively. The Class-0 clusters contain 2^t CQ_s 's and the Class-1 clusters contain 2^s CQ_t 's. They will be referred to as clusters of opposite class of each other, same class otherwise. The function f is a bijection between nodes of Class-0 clusters and those of Class-1 clusters such that, for u, v , two nodes of the same cluster, $f(u)$ and $f(v)$, are in two different clusters, and the edge $(u, f(u))$ is a cross edge. The bijection f ensures the existence of a perfect matching between two nodes in different clusters, but there is no requirement for the specifics of the perfect matching. Therefore, we have the following proposition.

Proposition 22. $GECQ(s, t)$ can be decomposed into two disjoint subgraphs \tilde{L}' and \tilde{R}' . And \tilde{L}' and \tilde{R}' are the subgraphs induced by $V(\tilde{L}')$ and $V(\tilde{R}')$, respectively, where

$$V(\tilde{L}') = \{a_{s-1}a_{s-2} \cdots a_0 b_{t-1} \cdots b_0 0 | a_j, b_i \in \{0, 1\}, \quad (17)$$

with $0 \leq j \leq s-1$ and $0 \leq i \leq t-1$).

$$V(\widetilde{R}') = \{a_{s-1}a_{s-2} \cdots a_0 b_{t-1} \cdots b_0 I \mid a_j, b_i \in \{0, 1\}, \quad (18)$$

with $0 \leq j \leq s-1$ and $0 \leq i \leq t-1$).

By Definition 21, \widetilde{L}' can be partitioned into 2^t subgraphs, denoted by \widetilde{L}_i' such that for $v_1, v_2 \in \widetilde{L}_i'$, $v_1[t:1] = v_2[t:1]$, where $i = 1, 2, \dots, 2^t$. Similarly, \widetilde{R}' can be partitioned into 2^s subgraphs, denoted by \widetilde{R}_j' such that $w_1, w_2 \in \widetilde{R}_j'$, $w_1[t+1:s+t] = w_2[t+1:s+t]$, for $j = 1, 2, \dots, 2^s$. And $GECQ(s, t)$ satisfies the following conditions (see $GECQ(1, 3)$ in Figure 7):

- (1) For any i, j , $\widetilde{L}_i' \cong CQ_s$ and $\widetilde{R}_j' \cong CQ_t$. Further, $|V(\widetilde{L}_i')| = 2^s$ and $|V(\widetilde{R}_j')| = 2^t$
- (2) Each node in $V(\widetilde{L}')$ has a sole neighbor in $V(\widetilde{R}')$ and vice versa. In addition, for distinct nodes in each \widetilde{L}_i' , their neighbors of \widetilde{R}' lie in different \widetilde{R}_j'
- (3) For any two different subgraphs \widetilde{L}_i' and \widetilde{L}_h' with $i \neq h$, there exists no edge between them. Similar for \widetilde{R}_j' and \widetilde{R}_k' with $j \neq k$.

By Proposition 22, the exchanged crossed cube $GECQ(s, t)$ is an exchanged X -cube, where the X -cube is a crossed cube. Then, the following theorems hold obviously.

Theorem 23. For any integers $1 \leq g \leq s-2$ and $3 \leq s \leq t$, $\kappa^g(GECQ(s, t)) = (s-g+1)2^g$.

Theorem 24.

- (1) For any integers $1 \leq g \leq s-2$ and $3 \leq s \leq t$, $t_g^P(GECQ(s, t)) = (s-g+2)2^g - 1$
- (2) For any integers $1 \leq g \leq s-2$ and $4 \leq s \leq t$, $t_g^M(GECQ(s, t)) = (s-g+2)2^g - 1$.

5.3. The Locally Generalized Exchanged Twisted Cube. The locally exchanged twisted cube proposed by Chang et al. [29], obtained by removing edges from a locally twisted cube LTQ_{s+t+1} . The definition of locally exchanged twisted cube is introduced as follows.

Definition 25 (see [29]). The (s, t) -dimensional locally exchanged twisted cube is defined as a graph $LETQ(s, t) = G(V(LETQ(s, t)), E(LETQ(s, t)))$ for $s, t \geq 1$. The node set $V(LETQ(s, t)) = \{x = x_{t+s} \cdots x_{t+1} x_t \cdots x_1 x_0 : x_i \in \{0, 1\} \text{ with } 0 \leq i \leq t+s\}$. $E(LETQ(s, t))$ is the edge set consisting of the following three types of disjoint sets E_1, E_2 , and E_3 .

$$\begin{aligned} E_1 &= \{(x, y) \in V \times V : x \oplus y = 2^0\} \\ E_2 &= \left\{ (x, y) \in V \times V : x_0 = y_0 = 1, x_1 = y_1 = 0 \text{ and } x \oplus y = 2^k \text{ for } k \in [3, t] \right\} \cup \left\{ (x, y) \in V \times V : x_0 = y_0 = x_1 = y_1 = 1 \text{ and } x \oplus y = 2^k + 2^{k-1} \text{ for } k \in [3, t] \right\} \\ &\quad \cup \left\{ (x, y) \in V \times V : x_0 = y_0 = 1 \text{ and } x \oplus y \in \{2^1, 2^2\} \right\} \\ E_3 &= \left\{ (x, y) \in V \times V : x_0 = y_0 = x_{t+1} = y_{t+1} = 0 \text{ and } x \oplus y = 2^k \text{ for } k \in [t+3, t+s] \right\} \\ &\quad \cup \left\{ (x, y) \in V \times V : x_0 = y_0 = 0, x_{t+1} = y_{t+1} = 1 \text{ and } x \oplus y = 2^k + 2^{k-1} \text{ for } k \in [t+3, t+s] \right\} \\ &\quad \cup \left\{ (x, y) \in V \times V : x_0 = y_0 = 0 \text{ and } x \oplus y \in \{2^{t+1}, 2^{t+2}\} \right\} \end{aligned} \quad (19)$$

Let $s, t \geq 1$; there are two classes of locally twisted cubes in the locally generalized exchanged twisted cube $LGETQ(s, t, f)$: one class, referred to as the Class-0 clusters, contains 2^t LTQ_s 's; and the other, referred to as the Class-1 clusters, contains 2^s LTQ_t 's. They will be referred to as clusters of opposite class of each other, same class otherwise. There exists a bijection function f between nodes of Class-0 clusters and those of Class-1 clusters. For two nodes u, v in the same cluster, $f(u)$ and $f(v)$ belong to two different ones, and the edge $(u, f(u))$ is a cross edge. The bijection f ensures the existence of a perfect matching between nodes of Class-0 clusters and those in the Class-1 clusters, but the specifics of the perfect matching can be ignored. Further, we obtain the proposition as follows.

Proposition 26. $LGETQ(s, t)$ can be decomposed into two disjoint subgraphs \widetilde{L}' and \widetilde{R}' . \widetilde{L}' can be partitioned into 2^t subgraphs, denoted by \widetilde{L}_i' for $i = 1, 2, \dots, 2^t$. Similarly, \widetilde{R}' can be partitioned into 2^s subgraphs, denoted by \widetilde{R}_j' for $j = 1, 2, \dots, 2^s$. And $LGETQ(s, t)$ satisfies the following conditions (see $LGETQ(1, 3)$ in Figure 8):

- (a) For any i, j , $\widetilde{L}_i' \cong LTQ_s$ and $\widetilde{R}_j' \cong LTQ_t$. Further, $|V(\widetilde{L}_i')| = 2^s$ and $|V(\widetilde{R}_j')| = 2^t$
- (b) Each node in $V(\widetilde{L}')$ has a sole neighbor in $V(\widetilde{R}')$ and vice versa. In addition, for distinct nodes in each \widetilde{L}_i' , their neighbors of \widetilde{R}' lie in different \widetilde{R}_j'
- (c) For any two different subgraphs \widetilde{L}_i' and \widetilde{L}_h' with $i \neq h$, there exists no edge connects them. Similar for \widetilde{R}_j' and \widetilde{R}_k' with $j \neq k$.

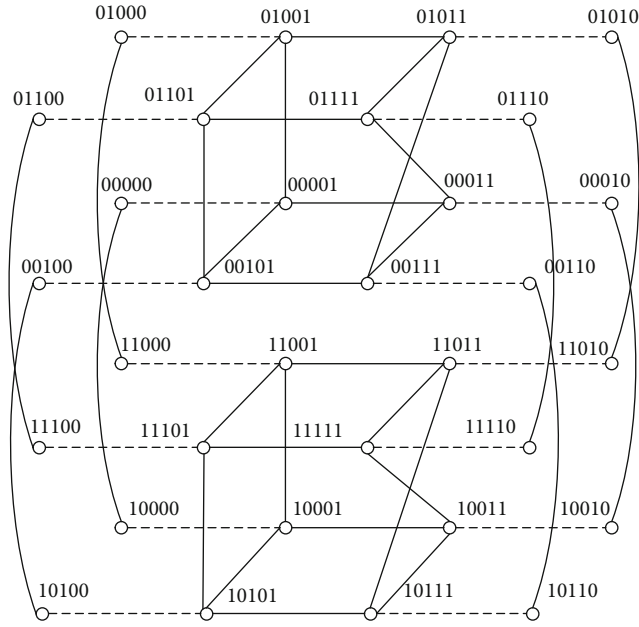


FIGURE 7: An exchanged crossed cube $GECQ(1, 3)$.

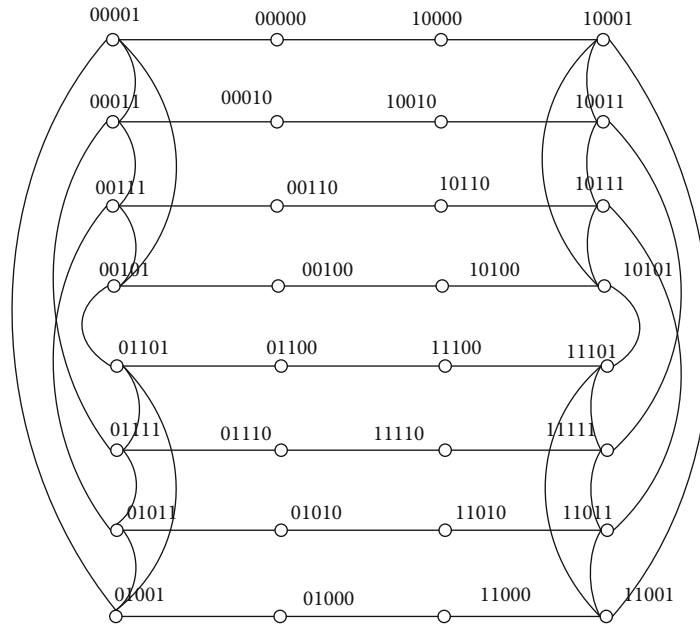


FIGURE 8: A locally exchanged twisted cube $LGETQ(1, 3)$.

By Proposition 26, the locally exchanged twisted cube $LGETQ(s, t)$ is a member of generalized exchanged X -cubes, where the X -cube is a locally twisted cube. Then, we have the following theorems.

Theorem 27.

- (1) For any integers $1 \leq g \leq s - 2$ and $3 \leq s \leq t$, $\kappa^g(LGETQ(s, t)) = (s - g + 1)2^g$
- (2) For any integers $1 \leq g \leq s - 2$ and $3 \leq s \leq t$, $t_g^P(LGETQ(s, t)) = (s - g + 2)2^g - 1$

- (3) For any integers $1 \leq g \leq s - 2$ and $4 \leq s \leq t$, $t_g^M(LGETQ(s, t)) = (s - g + 2)2^g - 1$.

6. Compare Results

In this section, we will illustrate the advantages of R^g -connectivity and g -GNCD compared to traditional connectivity and traditional diagnosability, respectively. Let us review their definition. The *connectivity*, which is less than the minimum degree of graph, is the minimum number of nodes that make the graph disconnected. The maximum number of faulty processors that the system can precisely point out is

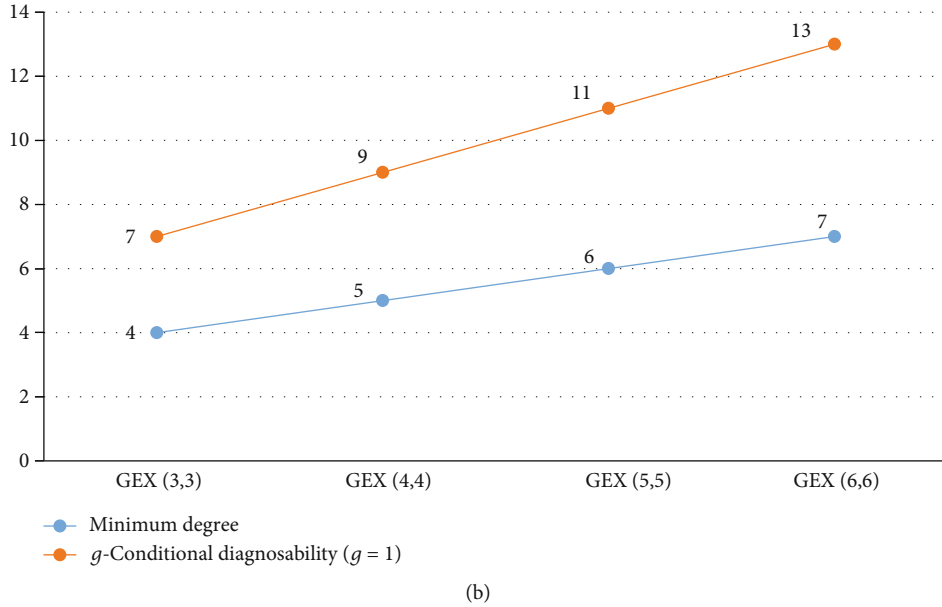
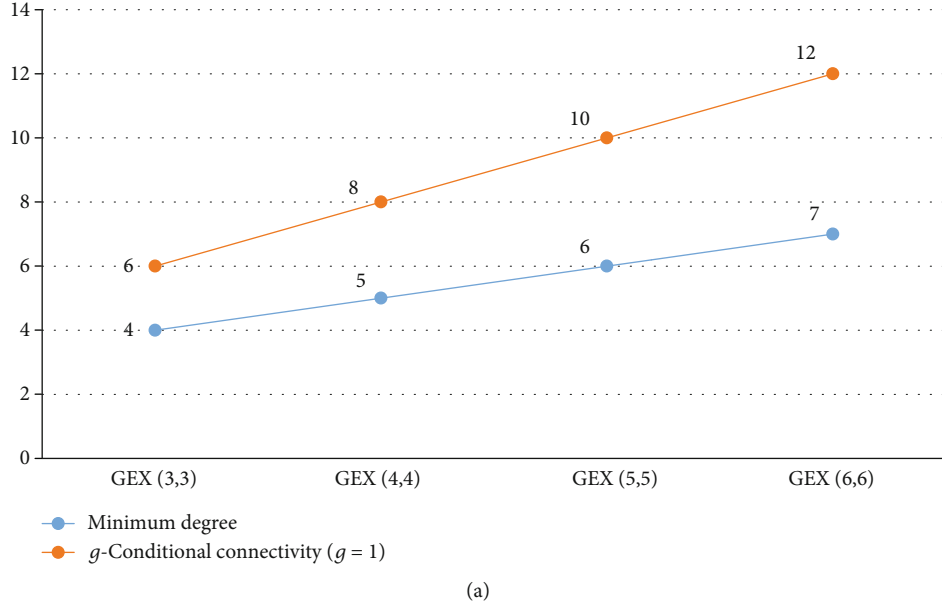


FIGURE 9: (a) The minimum degree and R^g -connectivity of $GEX(s, t)$. (b) The minimum degree and g -GNCD of $GEX(s, t)$.

as known as the *diagnosability* of the system, which is equal to the minimum degree of graph in most cases. The g -good-neighbor conditional connectivity (R^g -connectivity) is the minimum number of nodes that make the graph disconnected, and each node has at least g neighbors in every remaining component. The g -good-neighbor conditional diagnosability (g -GNCD) is the maximum number of faulty processors that can be identified under the condition that every fault-free processor has no less than g fault-free neighbors. We have determined that the R^g -connectivity of $GEX(s, t)$ is $(s - g + 1)2^g$ and the g -GNCD of $GEX(s, t)$ is $(s - g + 2)2^g - 1$. Figure 9 shows that R^g -connectivity and g -GNCD are both about 2^g times the minimum degree of graph. Therefore, we can speculate that R^g -connectivity is about 2^g times traditional connectivity and g -GNCD is about 2^g times traditional diagnosability, which means that R^g -

connectivity and g -GNCD can better evaluate the fault tolerance of network.

7. Conclusion

The R^g -connectivity and g -GNCD are two significant metrics for reliability of multiprocessor systems. Exchanged X -cubes are a class of irregular networks, obtained by deleting links from hypercubes and some variant networks of hypercubes (X -cubes). They not only combine the advantages of X -cubes but also reduce the interconnection complexity. Exchanged X -cubes classify its nodes into two different classes clusters with a unique connecting rule. In this paper, we propose the generalized exchanged X -cubes framework so that architecture can be constructed by different connecting rules. We first give the definition of a family of

generalized exchanged X -cubes, including generalized exchanged hypercubes, dual-cube-like networks, generalized exchanged crossed cubes, and locally exchanged twisted cubes as members. Then we determine the R^g -connectivity and g -GNCD of generalized exchanged X -cubes. Finally, the R^g -connectivity and g -GNCD of generalized exchanged hypercubes, dual-cube-like networks, generalized exchanged crossed cubes, and locally exchanged twisted cubes are established directly. As a future research, we attempt to evaluate the R^g -connectivity and g -GNCD of other generalized exchanged X -cubes using methods extended from the proposed method in this paper.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors have declared that no conflict of interest exists.

Acknowledgments

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