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Research Article

Second-Order Delay Differential Equations to Deal the Experimentation of Internet of Industrial Things via Haar Wavelet Approach

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In this article, an efficient numerical approach for the solution of second-order delay differential equations to deal with the experimentation of the Internet of Industrial Things (IIoT) is presented. With the help of the Haar wavelet technique, the considered problem is transformed into a system of algebraic equations which is then solved for the required results by using Gauss elimination algorithm. Some numerical examples for convergence of the proposed technique are taken from the literature. Maximum absolute and root mean square errors are calculated for various collocation points. The results show that the Haar wavelet method is an effective method for solving delay differential equations of second order. The convergence rate is also measured for various collocation points, which is almost equal to 2.

1. Introduction

Delay differential equations (DDEs) are type of DEs in which the solution of the unknown function is given in the previous time interval. A system whose performance does not depend directly on time is a time-invariant delay system. This systems can be defined by constant coefficients of the nth order ordinary DEs [1]. DDEs are used for modelling of various phenomenon such as modelling of systems with memory, electric circuits, and mechanical systems. The application of these systems in population dynamics [2] can be used in communication networks, mass transportation, remote controls, and biological systems. Many of the processes, both natural and artificial, in medicine, chemistry, engineering, economics, etc. involve time delays. The Internet of Things (IoT) contributes in facilitating the needs of daily life such as IoT for healthcare using effects of Mobile computing [3] and nonlinear delay integro-differential equations for wireless sensor network and IoT [4].

There are numerous approaches available in the literature for the solution DDEs of second order. Seong and Majid [1] developed the Adams Moulton technique to solve the second-order DDEs. Ibrahim [5], used 2h-step Spline method to solve the DDEs. Sedaghat [2], utilized the Chebyshev polynomials method to find the solution of DDEs. Ehigie et al. [6] implement a 3-point block technique to solve DDEs of second order. Chebyshev wavelet technique was developed by Ghasemi and Kajan [7] to solve the DDEs. Ahmad et al. [8] solved the DDEs by a block hybrid method. Multistep methods was used by Okunuga and Ehigie [9] to solve the DDEs. Brown [10] used a method of implicit multistep to solve the DDEs. Ismail et al. [11] found the solution of DDEs by 3-point block methods. Ehigie et al. [12] used a method of 2-step continuous linear multistep to solve the second-order DDEs. Some other well-known methods are the following: Runge-Kutta [13], Shift Walsh matrix method [14], Hermite interpolation method [15], method of retarded initial value problems [16], one-step collocation method [17],

coupled block technique [18], one-step block techniques [11], implicit block technique [19], Direct integration implicit variable method [19], predictor-corrector method [20], Taylor method [21], fuzzy mapping and control method [22], variational iteration method [23, 24], and Galerkin method [25]. A structure for the IIoT cloud-fog hybrid network for industry data processing was proposed by Liu et al. [26]. Sahal et al. [27] studied the strong point and flaws of open source technologies for big data. Khan et al. [28] offered the idea of IIoT in a novel manner for supporting readers to comprehend the IIoT. Gulati and Kaur [29] analysed the main opportunities assimilated from the idea of IoT into industry with suggesting reference architecture. The use of Haar wavelet have wide-ranging applications in scientific computing. The Haar Collocation Technique (HCT) is used for fractional Riccati type differential equations [30], Birthmark based identification [31], delay Fredholm-Volterra integral equations [32], delay integrodifferential equations [4], systems of fractional differential equations [33], and fractional integrodifferential equations [34] in recent literature. This article studies the solutions of second-order DDEs, that is, we develop numerical technique using Haar wavelet with constant delay.

In this paper, we discuss the solution of the second-order DDEs using a HCT to deal with the experimentation of IIoT, consider linear DDEs as

$$\begin{cases} w''(t) = a(t)w'(t) + b(t)w(t) + c(t)u(t) + d(t)w(t-\tau) + e(t)u(t-\tau), \\ w'(0) = \alpha_1, w(0) = \alpha_2, \\ w(t) = \phi(t), -\tau \le t < 0, \end{cases} \tag{1}$$

where $u(t)$ is a control function, $\phi(t)$ is the delay condition, and $w(t)$ is a state function.

The paper is organized as some basic results and notions are given in Section 2. Section 3 provides the HCT solution for linear DDEs of second order. In Section 4, the HCT validation is given. The results are discussed in Section 5, and the conclusion is given in the last part of the paper.

2. Haar Wavelet

Scaling function on $[\alpha_1, \alpha_2)$ is [35]

$$h_1(p) = \begin{cases} 1 & \text{for } p \in [\alpha_1, \alpha_2), \\ 0 & \text{elsewhere.} \end{cases} \tag{2}$$

Mother wavelet on $[\alpha_1, \alpha_2)$ is

$$h_2(p) = \begin{cases} 1 & \text{for } p \in \left[\alpha_1, \dfrac{\alpha_1+\alpha_2}{2}\right), \\ -1 & \text{for } p \in \left[\dfrac{\alpha_1+\alpha_2}{2}, \alpha_2\right), \\ 0 & \text{elsewhere.} \end{cases} \tag{3}$$

The other terms can be written as

$$h_i(p) = \begin{cases} 1 & \text{for } t \in [\eta_1, \eta_2), \\ -1 & \text{for } t \in [\eta_2, \eta_3), \\ 0 & \text{elsewhere,} \end{cases} \tag{4}$$

where $\eta_1 = \alpha_1 + (\alpha_2 - \alpha_1)(\zeta/d), \eta_2 = \alpha_1 + (\alpha_2 - \alpha_1)(\zeta + 0.5/d), \eta_3 = \alpha_1 + (\alpha_2 - \alpha_1)(\zeta + 1/d)$, where $d = 2^r$, and $\zeta = 0, 1, \cdots, d-1$. The number i can be calculated as $i = d + \zeta + 1$. If we take interval $[0, 1]$, then values of η_1, η_2, and η_3 are: $\eta_1 = \zeta/d$, $\eta_2 = 1/2 + \zeta/d$, $\eta_3 = 1 + \zeta/d$. Any member in $L^2[0, 1)$, is $u(p) = \sum_{k=1}^{\infty} \lambda_k h_k(t)$, we truncate this series is $u(t) \approx \sum_{k=1}^{N} \lambda_k h_k(t)$. Let

$$p_{i,1}(t) = \int_0^t h_i(x)dx,$$
$$P_{i,1}(p) = \begin{cases} p - \eta_1 & \text{for } p \in [\eta_1, \eta_2), \\ \eta_3 - p & \text{for } p \in [\eta_2, \eta_3), \\ 0 & \text{elsewhere.} \end{cases} \tag{5}$$

Also, $p_{i,2}$ is

$$p_{i,2}(p) = \int_0^p p_{i,1}(r)dr. \tag{6}$$

We obtain

$$P_{i,2}(p) = \begin{cases} \dfrac{1}{2}(p - \eta_1)^2 & \text{if } p \in [\eta_1, \eta_2), \\ \dfrac{1}{4m^2} - \dfrac{1}{2}(\eta_3 - p)^2 & \text{if } p \in [\eta_2, \eta_3), \\ \dfrac{1}{4m^2} & \text{if } p \in [\eta_3, 1), \\ 0 & \text{elsewhere.} \end{cases} \tag{7}$$

Generally,

$$P_{i,n}(p) = \int_0^p P_{i,n-1}(r)dr. \tag{8}$$

Thus, [9],

$$P_{i,n}(t) = \begin{cases} 0 & \text{if } t \in [0, \eta_1), \\ \dfrac{(t-\eta_1)^n}{n!} & \text{if } t \in [\eta_1, \eta_2), \\ \dfrac{[(t-\eta_1)^n - 2(\eta_1-\eta_2)^n]}{n!} & \text{if } t \in [\eta_2, \eta_3), \\ \dfrac{1}{n!}[(t-\eta_1)^n - 2(\eta_1-\eta_2)^n + (t-\eta_3)^n], & \text{if } t \in [\eta_3, 1). \end{cases} \tag{9}$$

The $[a_1, a_2]$ interval for HCT is discretized as

$$p_i = a_1 + (a_2 - a_1)\frac{i-0.5}{2M} \quad i = 1, 2, 3, \cdots, 2M. \tag{10}$$

The points defined in the above Eq. (10) are called collocation points (CPs).

3. Numerical Method

Here, we describe the proposed method for second-order DDEs to deal with the experimentation of IIoT. Let $w''(t) \in L^2[0, 1)$, then

$$w''(t) = \sum_{i=1}^{N} \lambda_i h_i(t). \tag{11}$$

Integrating Eq. (11) from 0 to t,

$$w'(t) - w'(0) = \sum_{i=1}^{N} \lambda_i p_{i,1}(t), \tag{12}$$

from Eq. (1), $w'(0) = \alpha_1$, so we get

$$w'(t) = \alpha_1 + \sum_{i=1}^{N} \lambda_i p_{i,1}(t). \tag{13}$$

Now, integrating Eq. (13) from 0 to t, the following relation yields:

$$w(t) - w(0) = \alpha_1(t) + \sum_{i=1}^{N} \lambda_i p_{i,2}(t), \tag{14}$$

from Eq. (1), $w(0) = \alpha_2$, so we have

$$w(t) = \alpha_2 + \alpha_1(t) + \sum_{i=1}^{N} \lambda_i p_{i,2}(t). \tag{15}$$

By putting Eq. (11), Eq. (13), and Eq. (15) in Eq. (1), we get

$$\begin{aligned} &\sum_{i=1}^{N} \lambda_i h_i(t) - a(t)\sum_{i=1}^{N} \lambda_i p_{i,1}(t) - b(t)\sum_{i=1}^{N} \lambda_i p_{i,2}(t) \\ &= \begin{cases} a(t)\alpha_1 + b(t)(\alpha_2 + \alpha_1(t)) + c(t)u(t) + d(t)\phi(t-\tau) + e(t)u(t-\tau), \text{for} t < 0, \\ a(t)\alpha_1 + b(t)(\alpha_2 + \alpha_1(t)) + c(t)u(t) + d(t)\left(\alpha_2 + \alpha_1(t-\tau) + \sum_{i=1}^{N} \lambda_i p_{i,2}(t)(t-\tau)\right) + e(t)u(t-\tau), \text{for} t > 0. \end{cases} \end{aligned} \tag{16}$$

Discretizing this Eq. (16) at t_j CPs, we have

$$\begin{aligned} &\sum_{i=1}^{N} \lambda_i h_i(t_j) - a(t_j)\sum_{i=1}^{N} \lambda_i p_{i,1}(t_j) - b(t_j)\sum_{i=1}^{N} \lambda_i p_{i,2}(t_j) \\ &= \begin{cases} a(t_j)\alpha_1 + b(t_j)(\alpha_2 + \alpha_1(t_j)) + c(t_j)u(t_j) + d(t_j)\phi(t_j-\tau) + e(t_j)(u(t_j)-\tau), \text{for } t_j < 0 \\ a(t_j)\alpha_1 + b(t_j)(\alpha_2 + \alpha_1(t_j)) + c(t_j)u(t_j) + d(t_j)\left((\alpha_2 + \alpha_1(t_j-\tau)) + \sum_{i=1}^{N} \lambda_i p_{i,2}(t_j-\tau)\right) + e(t_j)u(t_j-\tau), \text{for } t_j > 0. \end{cases} \end{aligned} \tag{17}$$

The above system in matrix notations as given by

$$M\lambda = B, \tag{18}$$

where

$$\begin{aligned} M &= \left[m_{ij}\right]_{N\times N}, \lambda = \left[\lambda_j\right]_{N\times 1}, B = \left[b_j\right]_{N\times 1}, \\ b_j &= \begin{cases} a(t_j)\alpha_1 + b(t_j)(\alpha_2 + \alpha_1 t_j) + c(t_j)u(t_j) + d(t_j)\phi(t_j - \tau) + e(t_j)u(t_j - \tau) \text{ for } t_j < 0, \\ a(t_j)\alpha_1 + b(t_j)(\alpha_2 + \alpha_1 t_j) + c(t_j)u(t_j) + d(t_j)(\alpha_2 + \alpha_1(t_j - \tau) + e(t_j)u(t_j - \tau) \text{ for } t_j > 0, \end{cases} \\ m_{ij} &= \begin{cases} h_i(t_j) - a(t_j)p_{i,1}(t_j) - b(t_j)p_{i,2}(t_j) \text{ for } t_j < 0, \\ h_i(t_j) - a(t_j)p_{i,1}(t_j) - b(t_j)p_{i,2}(t_j) - d(t_j)p_{i,2}(t_j - \tau) \text{ for } t_j > 0, \end{cases} \end{aligned} \tag{19}$$

Hence, λ_i is calculated as $\lambda = M^{-1}B$. This is a linear system of order $N \times N$, which is solved by the Gauss elimination technique. By putting these λ_i's in Eq. (15), we get the required solution of second-order DDEs defined in (1).

4. Numerical Examples

Let w_{ap} be approximate and w_{ex} is exact solution for CPs and GPs, then maximum absolute L_{cp} error is $L_{cp} = \max |w_{exc} - w_{apc}|$. The M_{cp} root mean square errors at CPs is $M_{cp} = \sqrt{(1/N)(\Theta|w_{exc} - w_{apc}|^2)}$. In CPs, convergence rate R_{cp} is $R_{cp} = \log [w_{apc}(N/2)/w_{apc}(N)]/\log 2$.

Example 1. Consider DDE of second order [8]

$$w''(t) = -\frac{1}{2}w(t) + \frac{1}{2}w(t - \pi), 0 \le t \le 8\pi, \tag{20}$$

with delay condition

$$\mathrm{w(t) = sint}, -\pi \le \mathrm{t} \le 0, \tag{21}$$

and initial condition

$$\mathrm{w(0) = 0, w'(0) = 1}. \tag{22}$$

The analytical solution is $w(t) = \sin t$.

Example 2. Consider the following second-order DDE [1]

$$w''(t) = -\frac{1}{2}w(t) + \frac{1}{2}w(t - \pi), t \in [0, \pi], \tag{23}$$

with delay condition

$$w(t) = 1 - \sin t, -\pi \le t \le 0, \tag{24}$$

and initial condition

$$w'(0) = -1, w(0) = 1. \tag{25}$$

The exact solution is $w(t) = 1 - \sin t$.

Example 3. Consider the following second-order DDE [1]

$$w''(t) = w(t - \pi), t \in [0, \pi], \tag{26}$$

with delay condition

$$w(t) = \sin t, -\pi \le t \le 0, \tag{27}$$

and initial condition

$$w'(0) = 1, w(0) = 0. \tag{28}$$

The exact solution is $w(t) = \sin t$.

Example 4. Consider DDE of second order as [36]

$$w''(t) = -w(t) + w(t - 1), 0 \le t \le 2, \tag{29}$$

with delay condition

$$w(t) = t^2 + 3t + 2, -1 \le t \le 0, \tag{30}$$

and initial condition

$$w(0) = 2, w'(0) = 0. \tag{31}$$

The exact solution is $w(t) = t^2 + t - 2 + 4\cos t - \sin t, 0 \le t \le 1$.

Example 5. Consider the following second-order DDE [8]

$$w''(t) = -\frac{\sin t}{2 - \sin t}w(t - \pi), 0 \le t \le 8\pi, \tag{32}$$

Table 1: L_{cp}, $R_c(N)$, M_{cp}, and CPU time (seconds) for Example 1.

J	$N = 2^{J+1}$	L_{cp}	$R_c(N)$	M_{cp}	$R_c(N)$	CPU time (seconds)
1	4	3.97951×10^{-03}		2.69531×10^{-03}		0.00158
2	8	1.04840×10^{-03}	1.9326	6.78403×10^{-04}	1.92440	0.00193
3	16	2.68212×10^{-04}	1.9563	1.69886×10^{-04}	1.96674	0.00461
4	32	6.77789×10^{-05}	1.9850	4.24894×10^{-05}	1.98446	0.01682
5	64	1.70330×10^{-05}	1.9936	1.06234×10^{-05}	1.99250	0.06525
6	128	4.26915×10^{-06}	1.9966	2.65593×10^{-06}	1.99631	0.20768
7	256	1.06864×10^{-06}	2.0068	6.63892×10^{-07}	1.99817	0.81604
8	512	2.67328×10^{-07}	1.9892	1.65997×10^{-07}	1.99909	3.27512
9	1024	7.68531×10^{-08}	1.7977	4.14994×10^{-08}	1.79843	13.0245

Table 2: L_{cp}, $R_c(N)$, M_{cp}, and CPU time (seconds) for Example 2.

J	$N = 2^{J+1}$	L_{cp}	$R_c(N)$	M_{cp}	$R_c(N)$	CPU time (seconds)
1	4	3.97951×10^{-03}		2.69531×10^{-03}		0.00158
2	8	1.04840×10^{-03}	1.9326	6.78403×10^{-04}	1.92440	0.00193
3	16	2.68212×10^{-04}	1.9563	1.69886×10^{-04}	1.96674	0.00461
4	32	6.77789×10^{-05}	1.9850	4.24894×10^{-05}	1.98446	0.01682
5	64	1.70330×10^{-05}	1.9936	1.06234×10^{-05}	1.99250	0.06525
6	128	4.26915×10^{-06}	1.9966	2.65593×10^{-06}	1.99631	0.20768
7	256	1.06864×10^{-06}	2.0068	6.63892×10^{-07}	1.99817	0.81604
8	512	2.67328×10^{-07}	1.9892	1.65997×10^{-07}	1.99909	3.27512
9	1024	7.68531×10^{-08}	1.7977	4.14994×10^{-08}	1.79843	13.0245

Table 3: L_{cp}, $R_c(N)$, M_{cp}, and CPU time (seconds) for Example 3.

J	$N = 2^{J+1}$	L_{cp}	$R_c(N)$	M_{cp}	$R_c(N)$	CPU time (seconds)
1	4	4.26909×10^{-03}		2.83925×10^{-03}		0.00140
2	8	1.13460×10^{-03}	1.9145	7.14201×10^{-04}	1.91174	0.00166
3	16	2.91783×10^{-04}	1.9572	1.78823×10^{-04}	1.95921	0.00334
4	32	7.39440×10^{-05}	1.9774	4.47228×10^{-05}	1.98039	0.01211
5	64	1.86096×10^{-05}	1.9903	1.11817×10^{-05}	1.99038	0.03886
6	128	4.66777×10^{-06}	1.9969	2.79551×10^{-06}	1.99524	0.14378
7	256	1.16886×10^{-06}	2.0062	6.98881×10^{-07}	1.99763	0.53690
8	512	2.92454×10^{-07}	1.9901	1.74720×10^{-07}	1.99882	2.59874
9	1024	7.31435×10^{-08}	1.9980	4.36801×10^{-08}	1.99940	11.0023

with delay condition

$$w(t) = 2 + \sin t - \pi \leq t \leq 0, \tag{33}$$

and initial condition

$$w'(0) = 1, w(0) = 2. \tag{34}$$

The exact solution is $w(t) = 2 + \sin t$.

5. Discussion

The second-order derivative in DDE is expressed as Haar function and the value of the first derivative is obtained by the process of integration. By applying the HCT, we obtain a system of linear equations by substituting CPs. The method of Gauss elimination is used for this system. Finally, by utilizing these coefficients, the solution at CPs is obtained. L_{cp} and M_{cp} errors for different number of CPs are given in Tables.

Table 4: L_{cp}, $R_c(N)$, M_{cp}, and CPU time (seconds) for Example 4.

J	$N = 2^{J+1}$	L_{cp}	$R_c(N)$	M_{cp}	$R_c(N)$	CPU time (seconds)
1	4	7.24881×10^{-03}		4.60001×10^{-03}		0.00197
2	8	1.95251×10^{-03}	1.8925	1.14978×10^{-03}	2.01260	0.00277
3	16	5.05791×10^{-04}	1.9491	2.87450×10^{-04}	1.98991	0.04996
4	32	1.28646×10^{-04}	1.9801	7.18631×10^{-05}	1.99899	0.01505
5	64	3.24351×10^{-05}	1.9821	1.79658×10^{-05}	2.00402	0.05394
6	128	8.14288×10^{-06}	1.9929	4.49145×10^{-06}	1.99517	0.19726
7	256	2.03997×10^{-06}	2.0035	1.12286×10^{-06}	2.00321	0.77262
8	512	5.10526×10^{-07}	1.9929	2.80716×10^{-07}	2.00000	3.07790
9	1024	1.27698×10^{-07}	2.0057	7.01790×10^{-08}	1.99794	12.3348

Table 5: L_{cp}, $R_c(N)$, M_{cp}, and CPU time (seconds) for Example 5.

J	$N = 2^{J+1}$	L_{cp}	$R_c(N)$	M_{cp}	$R_c(N)$	CPU time (seconds)
1	4	4.26909×10^{-03}		2.83925×10^{-03}		0.00196
2	8	1.13460×10^{-03}	1.9145	7.14201×10^{-04}	1.9868	0.00301
3	16	2.91783×10^{-04}	1.9572	1.78823×10^{-04}	2.0040	0.00947
4	32	7.39440×10^{-05}	1.9774	4.47228×10^{-05}	1.9935	0.02115
5	64	1.86096×10^{-05}	1.9903	1.11817×10^{-05}	2.0097	0.06661
6	128	4.66777×10^{-06}	1.9969	2.79551×10^{-06}	1.9922	0.25637
7	256	1.16886×10^{-06}	2.0062	6.98881×10^{-07}	1.9990	1.03078
8	512	2.92454×10^{-07}	1.9901	1.74720×10^{-07}	2.0041	4.09742
9	1024	7.31435×10^{-08}	1.9980	4.36801×10^{-08}	1.9967	16.4587

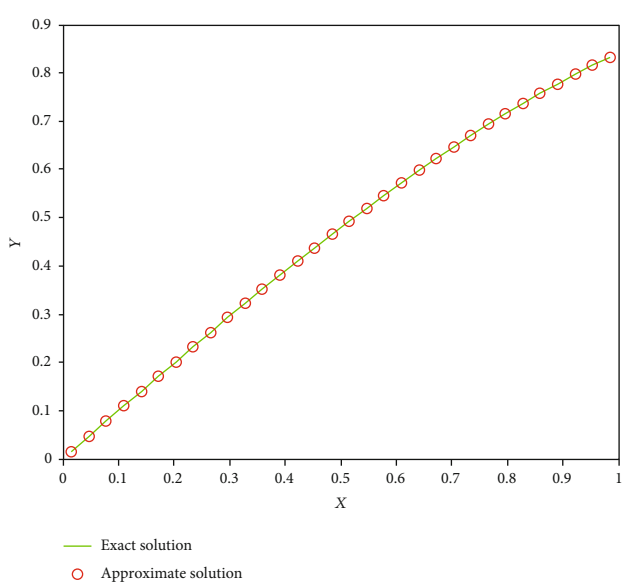

Figure 1: Comparison of both exact and approximate solution for 32 CPs for Example 1.

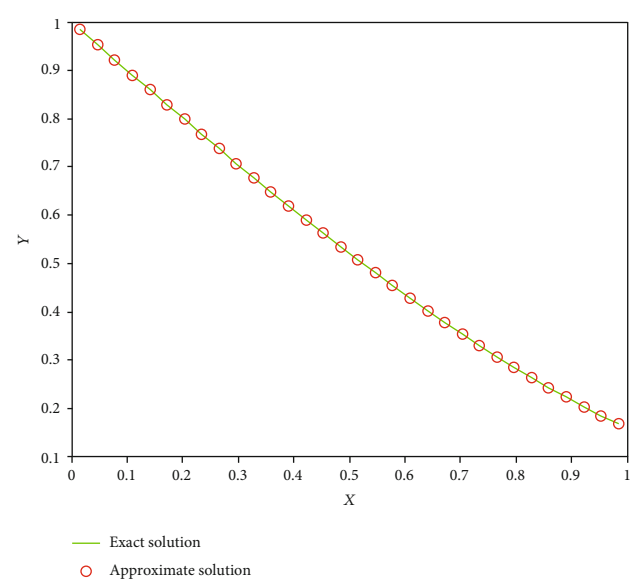

Figure 2: Comparison of both exact and approximate solution for 32 CPs for Example 2.

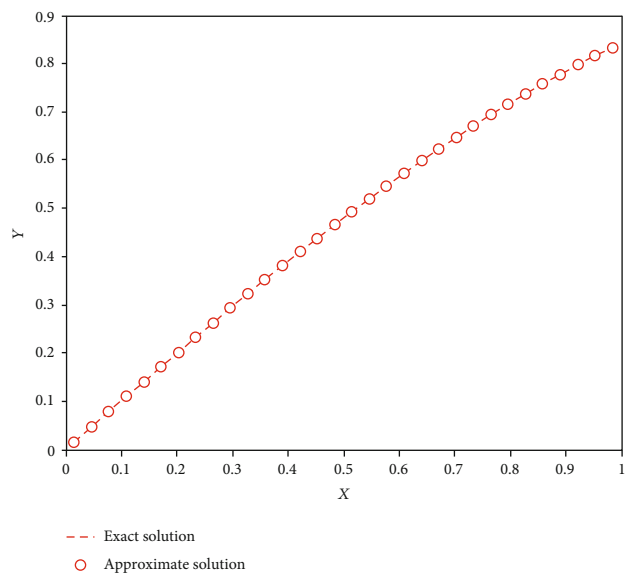

Figure 3: Comparison of both exact and approximate solution for 32 CPs for Example 3.

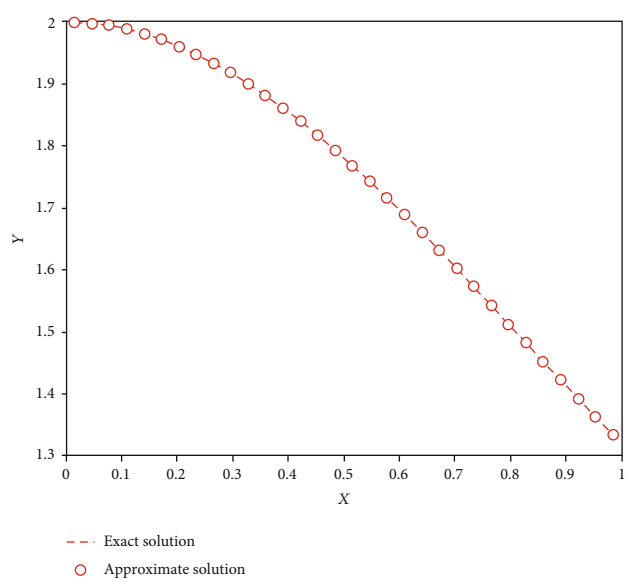

Figure 4: Comparison of both exact and approximate solution for 32 CPs for Example 4.

$R_c(N)$ and CPU time (seconds) are also reported in tables for each example. For Example 1, L_{cp} and M_{cp} errors for different number of CPs are shown in Table 1. Table 2 shows the errors for different number of CPs for Example 2, Table 3 represents errors for different number of CPs for Example 3, Table 4 shows the errors for different number of CPs for Example 4, and Table 5 shows the errors for different number of CPs for Example 5. All errors are decreased by taking more CPs. R_{cp} is determined which is nearly equal to 2, supporting the results shown in [37] by Majak et al. The comparison of both numerical and analytical solution at $N = 32$ CPs is also shown in figures. Figure 1 represents the comparison of approximate and exact solution for Example 1, Figure 2 represents the comparison of approximate and exact solution for

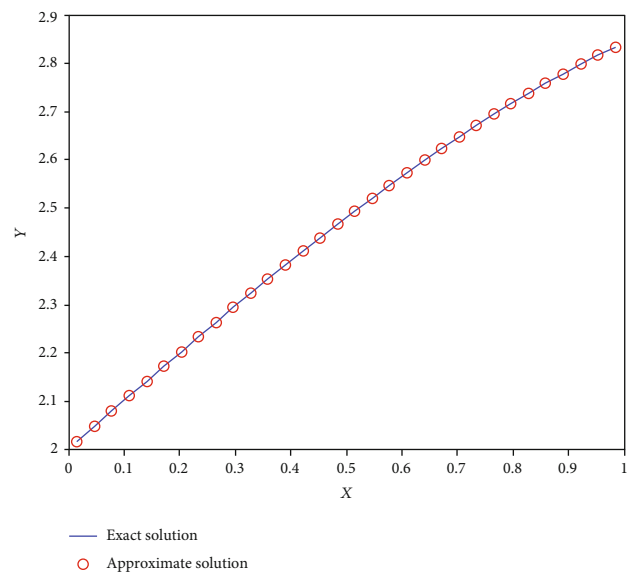

Figure 5: Comparison of both exact and approximate solution for 32 CPs for Example 5.

Example 2, Figure 3 represents the comparison of approximate and exact solution for Example 3, Figure 4 represents the comparison of approximate and exact solution for Example 4, and Figure 5 represents the comparison of exact and approximate solution for Example 5.

6. Conclusion

HCT scheme is devoted for the solution of second-order DDEs to deal with the experimentation of IIoT. The Haar technique is applied to linear DDEs for dealing with the experimentation of the Internet of Industrial Things. The Matlab software is utilized to experiment the Haar wavelet technique with the examples, and the numerical solution is compared with the exact solution. We compare the obtained solution with the exact solution and also we compute the L_{cp} and M_{cp} errors to show the accuracy of the Haar wavelet technique. We give some test problems for the illustration of our results. The experimental rate $R_c(N)$ of convergence for different number of CPs is also calculated which is approximately equal to 2. The results show that HCT is efficient for solving second-order DDEs. Our future work addresses to overcome the limitation of this study. Moreover, we will apply this technique to high order DDEs and system of DDEs.

Data Availability

No data is available.

Conflicts of Interest

There is no conflicting interest.

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