

Research Article

Incentives against Max-Min Fairness in a Centralized Resource System

Zheng Chen ¹, Zhaoquan Gu ², and Yuexuan Wang ³

¹College of Computer Science and Technology, Zhejiang University, Hangzhou, China

²Cyberspace Institute of Advanced Technology, Guangzhou University, Guangzhou, China

³College of Computer Science and Technology, Zhejiang University, Hangzhou, China

Correspondence should be addressed to Zhaoquan Gu; zqgu@gzhu.edu.cn

Received 9 January 2021; Revised 2 June 2021; Accepted 11 September 2021; Published 18 October 2021

Academic Editor: Lin Wang

Copyright © 2021 Zheng Chen et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Resource allocating mechanisms draw much attention from various areas, and exploring the truthfulness of these mechanisms is a very hot topic. In this paper, we focus on the max-min fair allocation in a centralized resource system and explore whether the allocation is truthful when a node behaves strategically. The max-min fair allocation enables nodes receive appropriate resources, and we introduce an efficient algorithm to find out the allocation. To explore whether the allocation is truthful, we analyze how the allocation varies when a new node is added to the system, and we discuss whether the node can gain more resources if it misreports its resource demands. Surprisingly, if a node misrepresents itself by creating several fictitious nodes but keeps the sum of these nodes' resource demands the same, the node can achieve more resources evidently. We further present some illustrative examples to verify the results, and we show that a node can achieve 1.83 times resource if it misrepresents itself as two nodes. Finally, we discuss the influence of node's misrepresenting behavior in tree graph: some child nodes gain fewer resources even if their parent node gains more resources by creating two fictitious nodes.

1. Introduction

Fairly allocating limited resources (such as fossil fuels, clean water, and cyber source) has been widely studied. With so many competing needs and demands, lots of relevant works have been done. In early 1940s, considering the notion of proportionality, Steinhaus put forward a method to ensure n participants receive at least $1/n$ of each participant's own value for getting everything [1]. In a 1958 book of mathematical, Gamow and Stern [2] introduce the formal notion of "envy-freeness," in which participants preferring to keep their own allocation to swapping with other participants. Since then, research in this area has aroused widespread concern and has been applied to solve many important real-world problems [3–5].

Motivated by sharing economy concepts [6], nodes in a multiagent system may take strategic behaviors to gain larger utility. In [7], it proves that designing a truthful mechanism that guarantees a market equilibrium outcome in general is impossible when an agent has the incentive to

misreport its information (such as its resource amount or demand). Some extant works also show that the truthfulness of resource allocating mechanisms cannot hold for a cheating node [8, 9]. However, the impossibility theorem proposed in [7] is not always correct for some restricted settings. For example, the proportional sharing mechanism is proved to be robust to agents' cheating, including an agent misreporting its resource or the connectivity with the rest of network [10, 11].

In this paper, we explore the truthfulness of resource allocation in a restricted setting. We consider a centralized resource system where only a central controller holds the resources that can be allocated to other nodes. Each node reports its resource demand to the controller who is in charge of allocating appropriate resources to the node. For example, each computer in a cluster of data center may require some resources while the central controller would make a fair resource allocation to all computers. In order to evaluate the fairness of a resource allocation, we adopt the concept of *max-min fairness* (a widely used concept in

networks, such as window flow central protocols [12], packer-switched network [13], and data network [14]) and explore whether the max-min fair allocation in the centralized resource system could be truthful if a node behaves strategically.

To begin with, we study the property of max-min fair allocation and introduce an efficient algorithm to find out the resource allocation mechanism. Afterwards, we explore how the nodes would be affected if a new node is connected to the controller. To study the truthfulness of the max-min fair allocation, we assume the new node could misreport its resource demand, and we analyze how this behavior changes the other nodes' allocation. In addition, we explore the effect of a novel splitting strategy that is proposed in recent work [15], where a node misrepresents itself by creating several fictitious agents but keeps the sum of the nodes' resource demand unchanged.

We summarize the contributions of the paper as follows:

- (1) We introduce a simple and straightforward algorithm to find out the max-min fair allocation in a centralized resource system
- (2) When a new node is connected to the controller, we analyze how much resource the new node could receive and how the other nodes' resource would be affected
- (3) We explore the truthfulness of the max-min fair allocation. If the new node misreports its resource demand, we show that it cannot gain more resources through this strategic behavior
- (4) We analyze the effect of node's misrepresenting behavior. By splitting itself into multiple nodes, a node could gain more resources in the allocation. Specifically, if a node is split into two nodes, it can achieve about two times resource compared to the allocation that it is honest under some special situations
- (5) We analyze the effect of node's misrepresenting behavior for its child nodes on tree graph. Resources are passed down from the root node layer by layer. By splitting itself into two nodes, its child node gains fewer resources in the allocation under some special situations even if it received more resources from its parent node

We also present some numerical examples to illustrate these interesting results, and the examples show that the splitting strategy could indeed help a node gain about 1.83 times resource if it is split into two nodes (as shown in Table 1). What is more, some numerical examples are presented to show the influence of parent's splitting strategy on its child on tree graph. Some children gain fewer resources by its parent's splitting strategy.

The rest of the paper is organized as follows. The next section lists some related works about truthfulness of resource exchanging and allocation mechanisms. The preliminaries are introduced in Section 3. We present the effi-

TABLE 1: The max-min allocation after v_δ is split into two nodes.

Node i	m_i	t_i	x_{cv_i}	$x_{cv_i} \cdot t_i$	u_i
v_1	3	5	3	15	3
v_2	4	5	4	20	4
v_3	5	10	5	50	5
v_4	7	5	7	35	7
v_5	8	5	$85/12 \approx 7.083$	425/12	85/12
v_6	9	5	$85/12 \approx 7.083$	425/12	85/12
v_{p_1}	8	1	$85/12 \approx 7.083$	85/12	85/12
v_{p_2}	9	1	$85/12 \approx 7.083$	85/12	85/12

cient algorithm that finds out the max-min fair allocation in Section 4. When a new node is connected to the controller, we analyze the change of the allocation in Section 5. The truthfulness of the allocation is discussed in Section 6 if the new node misreports its resource, while the splitting strategy that could help gain more resources is analyzed in Section 7. The effect of node's splitting strategy for its child nodes on tree is analyzed in Section 8. Finally, we present some numerical examples in Section 9 and conclude the paper in Section 10.

2. Related Work

The star network is generic and representative for many communication or economic networks. For example, the star network is commonly used in clustering of multityped heterogeneous networks [16], which extend the classical clustering on bityped heterogeneous networks [17]. Various networks have been mapped to the star networks [18], and we study the problem on the star network.

There are lots of researches about resource allocation in centralized system. One of the most critical resource allocating problems is the cake-cutting problem that is dividing a birthday cake among several children. During the World War II, Steinhaus, Knaster et al. [1] initiated the study of fair cake-cutting algorithms. How to help players achieve fair allocations is one of the most important strands in cake-cutting problem. Since then, a large number of related studies have emerged [5, 19, 20].

In resource exchange networks, max-min fairness is a simple and well-recognized approach to define a fair allocation in network [21–24], which has been widely adopted in economics [15, 25]. It is also used in ant colony algorithm for load balancing in cloud systems [26], and it is combined with convolutional neural networks for image classification [27].

In resource exchange networks, agents may be strategic, which makes the truthfulness of allocation mechanism being of paramount importance. In order to gain larger utility, an agent may misreport its true information, which may affect the market equilibrium. In [7], the authors prove that it is impossible to design a truthful mechanism that guarantees a market equilibrium outcome in general. In a Fisher market game with linear utility, agents can unilaterally deviate and

get a better payoff [8]. Even more specifically, agents may double its benefit by strategic behaviors [9]. Therefore, we also discuss about the truthfulness of the max-min fair allocation in a star network in this paper.

Generally, there are two types of agent strategic behaviors: misreporting its own resource amount and cheating on its connectivity with the rest of the network. By cheating on the connectivity, such as a missing edge or splitting a node into multiple nodes may affect the resource allocation mechanism. In [28], it examines the incentive problem in mechanism design in Internet applications or peer-to-peer systems when an agent misrepresents itself by creating several fictitious agents, which is also called false name bid. In [29], it measures how an agent gains if it deviates from its truthful behavior. In [30], it shows that agents' selfish behavior may contribute to the loss of social efficiency in Nash equilibrium in comparison to social optimality. In this paper, we adopt the splitting strategy to gain more resources in a star network.

3. Preliminaries

Consider a centralized resource system $G = (V, E)$ where a central node c holds the resources W_c that can be allocated to the other nodes $E \setminus \{c\} = \{v_1, v_2, \dots, v_n\}$. For any node $v_i, i \in [1, n]$, denote m_i as the largest amount of resource v_i can receive (we regard it as resource demand or resource limitation). Considering an allocation mechanism, denote x_{cv_i} as the amount of resources node v_i receives from the central node, $x_{cv_i} \leq m_i$; denote $t_i \geq 1$ as the resource transfer rate due to the link quality between the nodes. We assume that the resources would decay along the link, and thus, the central node actually allocates $x_{cv_i} \cdot t_i$ resources to node v_i (in traditional resource exchange or allocation mechanisms, we can set $t_i = 1$). Hence, $\sum_{i=1}^n x_{cv_i} \cdot t_i \leq W_c$. Denote u_i as the utility of node v_i after receiving the resources from the central node, which can be represented by $u_i = f(x_{cv_i})$ where $f(\cdot)$ is a monotone increasing function. For simplicity, we assume $u_i = x_{cv_i}$ in this paper and denote $U = \{u_1, u_2, \dots, u_n\}$ as the utility vector. To explore the effect of nodes' strategic behavior, we denote s_v as the reported strategy of node v , $\vec{s} = (s_{v_1}, s_{v_2}, \dots, s_{v_n})$ as the nodes' reported strategy profile and $\vec{s}_{-v} = (s_{v_1}, s_{v_2}, \dots, s_{v_{-1}}, s_{v_{+1}}, \dots, s_{v_n})$ as the profile without node v . We denote $U_v(s_v, \vec{s}_{-v})$ as the utility of node v when it reports its strategy as s_v . The problem is to explore the truthfulness of the centralized max-min fair resource allocation problem. First, we are to find a max-min fair allocation vector $\vec{x}_{cv} = \{x_{cv_1}, x_{cv_2}, \dots, x_{cv_n}\}$. To begin with, we introduce the property of max-min fairness.

Definition 1 (see [31]). A vector \vec{x} is a max-min fair vector on set χ if and only if for all $\vec{y} \in \chi$, if there exists $s \in \{1, 2, \dots, n\}$ such that $y_s > x_s$, there must exist $t \in \{1, 2, \dots, n\}$ such that $y_t < x_t < x_s$.

The definition of a max-min fair vector reveals that increasing some element of a max-min fair vector \vec{x} must

be at the expense of decreasing some smaller (or equal) element of the vector.

Definition 2 (see [32]). A set χ is max-min achievable if there exists a max-min fair vector on χ .

The following proposition shows the uniqueness property of the max-min fair vector. If we can find out a max-min fair vector, it is the unique one.

Proposition 3 (see [31]). *If a max-min fair vector exists on a set χ , then it is unique.*

Definition 4. A mechanism is truthful if no agent can benefit strictly from misreporting its amount of resource irrespective of what is reported by other agents.

Formally, given a node v with true strategy s_v^* and the profile $\vec{s} = (s_v, \vec{s}_{-v})$, it holds that for any reported strategy s_v ,

$$U_v(s_v, \vec{s}_{-v}) \leq U_v(s_v^*, \vec{s}_{-v}). \quad (1)$$

In this paper, we first design a mechanism that finds out the max-min fair utility vector in a centralized resource system. Then, we explore the effect of two strategic behavior: misreporting and misrepresenting. We formalize the problems as:

Problem 5. Considering a centralized resource system $G = (V, E)$ where the central node holds resource W_c that can be allocated to n nodes $\{v_1, v_2, \dots, v_n\}$. The problem is to design an allocation mechanism such that the utility vector $U = \{u_1, u_2, \dots, u_n\}$ is the max-min fair vector.

The problem is equivalent to finding out the max-min fair allocation vector $\vec{X}_{cv} = \{x_{cv_1}, x_{cv_2}, \dots, x_{cv_n}\}$. If one node v_i gets more resources through another allocation mechanism, another node v_j who receives fewer resources than v_i (i.e., $x_{cv_j} < x_{cv_i}$) would get resources fewer than x_{cv_j} in the new mechanism; this implies that increasing a node's resource of the max-min fair allocation would reduce a node's resource who originally receives fewer resources.

The max-min fair allocation mechanism could ensure the fairness among the nodes. When the system changes, such as a new node is connected to the central node, how many resources the new node should receive and will the other nodes be affected? This is the second problem we study. We formulate the problem as follows.

Problem 6. Considering a centralized resource system $G = (V, E)$ and the max-min fair allocation vector $\vec{X}_{cv}^* = \{x_{cv_1}^*, x_{cv_2}^*, \dots, x_{cv_n}^*\}$, if a new node v_δ with resource limitation m_δ and transfer rate t_δ is connected to the central node, find out the max-min fair allocation vector $\vec{X}_{cv} = \{x_{cv_1}, x_{cv_2}, \dots, x_{cv_n}, x_{cv_\delta}\}$ and how the nodes are affected.

In practice, some nodes may cheat to gain more resources. A simple cheating strategy is to misreport its resource limitation m_δ . We also analyze the truthfulness of the new node in Problem 6 and show that misreporting cannot help gain more resources (see Section 6). Therefore, we propose the splitting strategy to gain more resources as the following problem.

Problem 7. Considering a centralized resource system $G = (V, E)$, if a new node is connected to the central node with resource limitation m_δ and transfer rate t_δ , we denote the max-min fair allocation vector as $\vec{X}_{cv} = \{x_{cv_1}, x_{cv_2}, \dots, x_{cv_n}, x_{cv_\delta}\}$. If the node is assumed to be split into two nodes v_{np_1}, v_{np_2} with resource limitations m_{np_1}, m_{np_2} ($m_{np_1} + m_{np_2} = m_\delta$) and the same transfer rate t_δ , find out the max-min fair allocation and analyze whether the node can gain more resources ($x_{cv_{np_1}} + x_{cv_{np_2}}$) compared to the received resource of the original allocation (x_{cv_δ}).

4. The Max-Min Fair Allocation Algorithm

We present the max-min fair allocation algorithm as Algorithm 1. The input of the algorithm is the resource limitation of each node $\vec{m} = \{m_1, m_2, \dots, m_n\}$ and the resource transfer rates $\{t_1, t_2, \dots, t_n\}$. The algorithm is to find the max-min fair vector $\vec{U} = \{u_1, u_2, \dots, u_n\}$, which is equivalent to the allocation vector $\vec{X}_{cv} = \{x_{cv_1}, x_{cv_2}, \dots, x_{cv_n}\}$.

As depicted in Algorithm 1, \vec{m} is firstly sorted by increasing order (line 3) and denotes the corresponding nodes as $\{v_1', v_2', \dots, v_n'\}$ where $m_i' \leq m_j'$ for any $i < j$. The resources are allocated to nodes from v_1' to v_n' according to lines 5-9. There are two situations: if the resources are adequate enough (line 5), node v_i' can receive the resources that are up to its limitation m_i' (line 6), or it is set as line 8. We show that the correctness of the algorithm by verifying the allocation vector $\vec{X}_{cv} = \{x_{cv_1}, \dots, x_{cv_n}\}$ (corresponding to the utility vector $\vec{U} = \{u_1, u_2, \dots, u_n\}$) is the max-min fair vector.

Theorem 8. *The output vector of Algorithm 1 is the max-min fair vector.*

Proof. For simplicity, we assume the input vector \vec{m} is already ordered such that $v_i' = v_i$. We prove the theorem from two situations. \square

For the first case, suppose $W_c \geq \sum_{j=1}^n t_j m_j$, which implies the central node has enough resources to ensure all nodes' resource demands can be satisfied. Thus, $\vec{X}_{cv} = \{m_1, m_2, \dots, m_n\}$. According to the definition of max-min vector, any element of the \vec{X}_{cv} can not be larger; hence, \vec{X}_{cv} is the max-min fair vector.

For the second case, suppose $W_c < \sum_{j=1}^n t_j m_j$; there must exist $i \in [1, n]$ such that $\forall j < i, x_{cv_j} = m_j$, and $x_{cv_i} = ((W_c - \sum_{j=1}^{i-1} t_j m_j) / \sum_{j=i}^n t_j) < m_j$ since Algorithm 1 computes x_{cv_i} sequentially from $i = 1$ to $i = n$. Then, we derive the output of Algorithm 1 as

$$\vec{X}_{cv} = \left\{ m_1, \dots, m_{i-1}, \frac{W_c - \sum_{j=1}^{i-1} t_j m_j}{\sum_{j=i}^n t_j}, \dots, \frac{W_c - \sum_{j=1}^{i-1} t_j m_j}{\sum_{j=i}^n t_j} \right\}. \quad (2)$$

For any other allocation vector $\vec{X}_{cv}^* = \{x_{cv_1}^*, \dots, x_{cv_n}^*\}$, if there exists $s \in [1, n]$ such that $x_{cv_s}^* > x_{cv_s}$, for any $t \in [1, n]$ satisfying $x_{cv_t} < x_{cv_s}$, suppose $x_{cv_t}^* \geq x_{cv_t}$; then, we have $x_{cv_1}^* t_1 + \dots + x_{cv_n}^* t_n > x_{cv_1} t_1 + \dots + x_{cv_n} t_n = W_c$, which leads to a contradiction since the allocation exceeds the amount of all resources W_c . Therefore, there exists $t \in [1, n]$ such that $x_{cv_t}^* < x_{cv_t} < x_{cv_s}$. By Definition 1, \vec{X}_{cv} is the max-min fair vector.

Combining the two cases, the theorem holds.

From Proposition 3, the max-min fair vector is unique, and Algorithm 1 can find out the vector efficiently. We analyze the complexity of Algorithm 1 briefly. The complexity of sorting on line 3 is $O(n \log n)$, and the complexity of computing each x_{cv_i} in each step (lines 5-9) is $O(1)$. Thus, the complexity of the algorithm is $O(n \log n + n) = O(n \log n)$. In [32], the Max-Min Programming Algorithm (MP) is proposed to find the smallest element of the max-min fair vector by solving several linear programming. The complexity of MP algorithm is $O(n \text{LP}(n, m))$ where $\text{LP}(n, m)$ is the complexity of a linear program. Considering the problem we study in the paper, MP algorithm can be applied to find out the max-min allocation vector, but it works much slower than Algorithm 1. In the following parts, we adopt our algorithm to find out the max-min allocation mechanism when a new node is inserted or a node is split into multiple nodes.

5. Resource Allocation Varies for a New Node

In a centralized resource system $G = (V, E)$ with the max-min fair allocation vector $\vec{X}_{cv}^* = \{x_{cv_1}^*, x_{cv_2}^*, \dots, x_{cv_n}^*\}$, if a new node v_δ is connected to the central node whose resource limitation is m_δ and transfer rate is t_δ , denote the max-min fair allocation vector of the new system as $\vec{X}_{cv} = \{x_{cv_1}, x_{cv_2}, \dots, x_{cv_n}, x_{cv_\delta}\}$. We analyze the relationship between \vec{X}_{cv}^* and \vec{X}_{cv} , which implies how the nodes' resources change regarding to the new inserted node.

Let R be the spare resources under the max-min fair allocation vector \vec{X}_{cv}^* before the new node is inserted; we formulate it as

$$R = W_c - \sum_{j=1}^n x_{cv_j}^* t_j. \quad (3)$$

```

1: Input  $\vec{m} = \{m_1, m_2, \dots, m_n\}$ ,  $\vec{t} = \{t_1, t_2, \dots, t_n\}$ 
2: Output  $\vec{U} = \{u_1, u_2, \dots, u_n\}$ 
3: Sort  $\vec{m}$  by increasing order as  $\vec{m}' = \{m'_1, m'_2, \dots, m'_n\}$  where  $m'_i \leq m'_j, \forall i < j$ ; denote the corresponding nodes as  $\{v_1', v_2', \dots, v_n'\}$ ;
4: for  $i \leftarrow 1$  to  $n$  do
5:   if  $\sum_{j=1}^{i-1} t_j' \cdot m'_j + m'_i, \sum_{j=i}^n t_j' \leq W_c$  then
6:      $x_{cv_i'} = m'_i$ 
7:   else
8:     for  $k \leftarrow i$  to  $n$  do
9:        $x_{cv_k'} = W_c - \sum_{j=1}^{i-1} t_j' \cdot m'_j / \sum_{j=i}^n t_j'$ ,  $u_k' = x_{cv_k'}$ 
10:    end for
11:    break;
12:   end if
13:    $u_i' = x_{cv_i'}$ ;
14: end for

```

ALGORITHM 1: The max-min fair allocation algorithm.

For simplicity, we suppose the nodes' resource limitations satisfy $m_1 \leq m_2 \leq \dots \leq m_n$. We analyze how the max-min fair allocation changes when node v_δ is added to the system from two cases:

- (i) Case 1. $R \geq t_\delta m_\delta$
- (ii) Case 2. $0 \leq R < t_\delta m_\delta$

To begin with, we introduce a property of the max-min allocation mechanism which implies that the nodes with fewer received resources remain the same if a node with more resources does not change.

Property 9. In the max-min allocation, if there exists $k \in [1, n]$ such that $x_{cv_k} = m_k$, for all $i \in [1, k)$, $x_{cv_i} = m_i$.

Proof. Suppose there exists $i \in [1, k)$ satisfying $x_{cv_i} < m_i$, according to the allocation algorithm (Algorithm 1),

$$\sum_{j=1}^{i-1} t_j m_j + m_i \sum_{j=i}^n t_j > W_c, \quad (4)$$

since $i < k$, we can derive

$$\begin{aligned} \sum_{j=1}^{k-1} t_j m_j + m_k \sum_{j=k}^n t_j &= \sum_{j=1}^{i-1} t_j m_j + \sum_{j=i}^{k-1} t_j m_j + m_k \sum_{j=k}^n t_j \\ &\geq \sum_{j=1}^{i-1} t_j m_j + m_i \sum_{j=i}^n t_j > W_c, \end{aligned} \quad (5)$$

hence $x_{v_k} < m_k$, which makes a contradiction. \square

We analyze how the max-min fair allocation varies for the first case.

Theorem 10. If $R \geq t_\delta m_\delta$, the new allocation holds $x_{cv_\delta} = m_\delta$, and $x_{cv_i} = m_i$ does not change when the new node v_δ is inserted.

Proof. When $R \geq t_\delta m_\delta$, the spare resources are enough. It is obvious that $W_c \geq t_\delta m_\delta + \sum_{j=1}^n t_j m_j$. Suppose there exists i satisfying $m_{i-1} \leq m_\delta \leq m_k$ (if $m_\delta \leq m_1$, we let $m_0 = 0$, and such i exists), we derive

$$\sum_{j=1}^{i-1} t_j m_j + t_\delta m_\delta + m_\delta \sum_{j=i}^n t_j \leq t_\delta m_\delta + \sum_{j=1}^n t_j m_j \leq W_c. \quad (6)$$

According to lines 5-8 of Algorithm 1, $x_{cv_\delta} = m_\delta$. From Property 9, $x_{cv_j} = m_j$ when $1 \leq j \leq i-1$, $m_j \leq m_\delta$. For any $k \in [i, n]$ where $m_k \geq m_\delta$, we can compute

$$m_\delta t_\delta + \sum_{j=1}^{k-1} t_j m_j + m_k \sum_{j=k}^n t_j \leq m_\delta t_\delta + \sum_{j=1}^n t_j m_j \leq W_c. \quad (7)$$

Hence,

$$m_k \leq \frac{W_c - \sum_{j=1}^{k-1} t_j m_j - m_\delta t_\delta}{\sum_{j=k}^n t_j}, \quad (8)$$

from lines 5-8 of Algorithm 1, $x_{cv_k} = m_k$. The proof for the special cases $m_\delta \leq m_1$ and $m_\delta \geq m_n$ is similar, and we omit the details. Combining these aspects, the theorem holds. \square

We then analyze how the max-min fair allocation varies for the second case when $0 \leq R < t_\delta m_\delta$. First, we find out a threshold r such that the nodes with resources fewer than r do not change their received resources (Lemma 11). Then, we analyze the change of nodes who received resources larger than r from two cases: $m_\delta > r$ (Lemma 12) and $m_\delta \leq$

r (Lemma 13). Finally, we compute the allocated resources of the new inserted node v_δ (Lemma 14).

Lemma 11. *If $R < t_\delta m_\delta$, there exists $r = (W_c - \sum_{j=1}^{i-1} t_j m_j) / \sum_{j=i}^n t_j + t_\delta$ such that any node $v_i, i \in [1, n]$ satisfying $m_i \leq r$, $x_{cv_i} = m_i$ under the new max-min fair allocation vector.*

Proof. Suppose $m_1 \leq m_2 \leq \dots \leq m_{i-1} \leq r \leq m_i \leq \dots \leq m_n$, we derive

$$\sum_{j=1}^{i-2} t_j m_j + m_{i-1} \sum_{j=i-1}^n t_j + t_\delta m_{i-1} \leq W_c, \quad (9)$$

hence

$$m_{i-1} \leq \frac{W_c - \sum_{j=1}^{i-2} t_j m_j}{t_\delta + \sum_{j=i-1}^n t_j}. \quad (10)$$

Then, we find the largest integer $i-1$ satisfying the above inequality, and we derive

$$r \left(\sum_{j=i}^n t_j + t_\delta \right) = W_c - \sum_{j=1}^{i-1} t_j m_j. \quad (11)$$

Thus, we figure out the threshold value r as

$$r = \frac{W_c - \sum_{j=1}^{i-1} t_j m_j}{\sum_{j=i}^n t_j + t_\delta}. \quad (12)$$

Since $x_{cv_{i-1}} = m_{i-1}$, for any node v_k with $m_k \leq m_{i-1} \leq r$, $x_{cv_k} = m_k$ by property 5. The lemma holds. \square

It is easy to check that the node v_k with $m_k \leq r$ received resources $x_{cv_k}^* = m_k$ in the original allocation vector before the new node is inserted. Finding out the threshold value r , these nodes' received resources do not change. For the nodes v_k with $m_k < r$, we analyze them from two cases: $m_\delta > r$ and $m_\delta \leq r$.

Lemma 12. *If $R < t_\delta m_\delta$, $m_\delta > r = (W_c - \sum_{j=1}^{i-1} t_j m_j) / \sum_{j=i}^n t_j + t_\delta$, for any node $v_i, i \in [1, n]$ satisfying $m_i > r$, $x_{cv_i} < x_{cv_i}^* \leq m_i$ under the new max-min fair allocation vector.*

Proof. First, we show that $x_{cv_i} < m_i$ under the new max-min fair allocation for any node v_i satisfying $m_i > r$. This means the threshold r divides the nodes into two subsets; one subset contains nodes that receive resources up to its limitation as Lemma 11, while the other subset consists of nodes with resource limitation larger than r and cannot receive resources up to its limitation.

Suppose there exists a node v_k such that $m_k > r$ and $x_{cv_k} = m_k$. According to Property 9, node v_i ($v \in [1, k-1]$) receives resources $x_{cv_i} = m_i$.

There are two cases: $m_k \geq m_\delta$ and $m_k < m_\delta$. For the first case that $m_k \geq m_\delta$, we drive

$$\sum_{j=1}^{k-1} t_j m_j + m_\delta t_\delta + m_k \sum_{j=k}^n t_j > \sum_{j=1}^{k-1} t_j m_j + r t_\delta + r \sum_{j=k}^n t_j = W_c, \quad (13)$$

which is a contradiction. When $m_\delta > m_k$, we can deduce a contradiction similarly. Therefore, $x_{cv_i} < m_i$ for node v_i with $m_i > r$.

Then, we show that node v_i with $m_i > r$ receives fewer resources compared to the original max-min fair allocation; that is, $x_{cv_i} < x_{cv_i}^*$. Denote $x_{cv_i}^* = \alpha$, suppose $x_{cv_i} \geq x_{cv_i}^* = \alpha$, we can derive

$$\sum_{j=1}^{i-1} t_j m_j + m_\alpha t_\delta + \alpha \sum_{j=i}^n t_j > \sum_{j=1}^{i-1} t_j m_j + \alpha \sum_{j=i}^n t_j > W_c, \quad (14)$$

which is a contradiction.

Combining the two sides, $x_{cv_i} < x_{cv_i}^* \leq m_i$ holds. \square

Lemma 12 implies that if the new node's resource limitation is large $m_\delta > r$, all nodes with large resources limitation ($m_i > r$) have to reduce its received resources. If $m_\delta \leq r$, fewer nodes have to reduce its resources, and we show it in the next lemma.

Lemma 13. *If $R < t_\delta m_\delta$, $m_\delta \leq r = (W_c - \sum_{j=1}^{i-1} t_j m_j) / \sum_{j=i}^n t_j + t_\delta$, there exists $r_1 \geq r$ such that the following two results hold:*

(i) *If node v_i satisfies $m_i > r_1$, $x_{cv_i} < x_{cv_i}^* \leq m_i$*

(ii) *If node v_i satisfies $m_i \leq r_1$, $x_{cv_i} = x_{cv_i}^* = m_i$*

Proof. If $m_\delta \leq r$, there exists $i_1 \leq i_2$ such that $m_1 \leq m_2 \leq \dots \leq m_{i_1-1} \leq m_\delta \leq \dots \leq m_{i_2-1} \leq r \leq m_{i_2} \leq \dots \leq m_n$; we have

$$\sum_{j=1}^{i_2-1} t_j m_j + m_\delta t_\delta + m_{i_2-1} \sum_{j=i_2}^n t_j \leq \sum_{j=1}^{i_2-1} t_j m_j + r \left(t_\delta + \sum_{j=i_2}^n t_j \right) \leq W_c, \quad (15)$$

thus $x_{cv_{i_2-1}} = m_{i_2-1}$. According to Property 9, $x_{cv_k} = m_k$ if $k \leq i_2 - 1$, which implies $x_{cv_\delta} = m_\delta$. Since $R < t_\delta m_\delta$, nodes with more resources would reduce its resources, and there must exist $r_1 \geq m_{i_2-1}$ such that $x_{cv_k} = x_{cv_k}^* = m_k$ if $m_k \leq r_1$ (nodes do not change their resources), and $x_{cv_k} < x_{cv_k}^* \leq m_k$ if $m_k > r_1$ (nodes reduce their resources).

Next, we show that the value $r_1 \geq r$. It is clear that if $r_1 \geq m_{i_2}$, $r_1 > r$. Suppose $m_{i_2-1} \leq r_1 < m_{i_2}$, we derive

$$r_1 \sum_{j=i_2}^n t_j + m_\delta t_\delta = W_c - \sum_{j=1}^{i_2-1} t_j m_j, \quad (16)$$

combining the formulation of r , we show that

$$(r_1 - r) \sum_{j=i_2}^n t_j = (r - m_\delta) t_\delta \geq 0. \quad (17)$$

Thus, $r_1 \geq r$ holds. It is easy to check node v_k with $m_k \leq r_1$ receives resources $x_{cv_k}^* = m_k$ in the original allocation. Finding out the threshold value r_1 , these nodes' received resources do not change. Under the new max-min fair allocation, the node v_k with $m_k > r_1$ receives resources $x_{cv_k} < x_{cv_k}^* \leq m_k$, and the proof is similar to that of Lemma 12. \square \square

Lemma 14. *If $R < t_\delta m_\delta$, there exists $r = (W_c - \sum_{j=1}^{i-1} t_j m_j) / \sum_{j=i}^n t_j + t_\delta$ such that $x_{cv_\delta} = m_\delta$ if $m_\delta \leq r$, and $x_{cv_\delta} < m_\delta$ if $m_\delta > r$.*

Proof. According to the proof of Lemma 13, $x_{cv_\delta} = m_\delta$ if $m_\delta \leq r$. We show that $x_{cv_\delta} < m_\delta$ if $m_\delta > r$.

Suppose $m_1 \leq m_2 \leq \dots \leq m_{i-1} \leq r \leq \dots \leq m_{i-1} \leq m_\delta \leq m_{i_2} \leq \dots \leq m_n$, if $x_{cv_\delta} = m_\delta$, we derive

$$\sum_{j=1}^{i_2-1} t_j m_j + m_\delta t_\delta + m_\delta \sum_{j=i_2}^n t_j > \sum_{j=1}^{i_2-1} t_j m_j + r t_\delta + r \sum_{j=i}^n t_j = W_c, \quad (18)$$

which is a contradiction. Thus, the lemma holds. \square \square

Combining Lemmas 11, 12, 13, and 14, we conclude the situation when the spare resources are inadequate $0 \leq R < t_\delta m_\delta$ as Theorem 15.

Theorem 15. *If $0 \leq R < t_\delta m_\delta$, there exist two thresholds r, r_1 as Lemma 11 and Lemma 13. If $m_\delta > r$, the nodes satisfy:*

$$\begin{cases} x_{cv_i} = x_{cv_i}^* & \text{if } m_i \leq r, \\ x_{cv_i} < x_{cv_i}^* \leq m_i & \text{if } m_i > r, \\ x_{cv_\delta} < m_\delta. \end{cases} \quad (19)$$

If $m_\delta \leq r$, the nodes satisfy:

$$\begin{cases} x_{cv_i} = x_{cv_i}^* & \text{if } m_i \leq r_1, \\ x_{cv_i} < x_{cv_i}^* \leq m_i & \text{if } m_i > r_1, \\ x_{cv_\delta} = m_\delta. \end{cases} \quad (20)$$

6. Truthfulness against the Misreporting Strategy

In this section, we analyze the truthfulness of the max-min fair allocation. Denote $\vec{X}_{cv} = (x_{cv_1}, x_{cv_2}, \dots, x_{cv_{i-1}}, x_{cv_\delta}, x_{cv_i}, x_{cv_n})$ as the allocation after the new node v_δ is inserted. Denote $\vec{M} = \{m_1, m_2, \dots, m_{i-1}, m_\delta, m_i, \dots, m_n\}$ as the resource limitation vector where the new node's limitation suits $m_{i-1} \leq m_\delta \leq m_i$.

Denote $\vec{M}_{-u} = \{m_1, m_2, \dots, m_n\}$ as the vector before the new node is inserted. We study how the allocation varies when the new node misreports its resource limitation.

As shown in the previous section, the allocated resources x_{cv_δ} of the new node v_δ can be regarded as a function of m_δ when \vec{M}_{-u} is fixed. Suppose the new node misreports its resource limitation as m_{nt} ; we analyze the effect from two situations $R \geq t_\delta m_\delta$ (the spare resources are adequate) or $R < t_\delta m_\delta$.

Lemma 16. *When $R \geq t_\delta m_\delta$, if the new node reports fewer resources ($m_{nt} \in (0, m_\delta]$), $x_{cv_\delta} = m_{nt}$; if the node reports more resources ($m_{nt} \in (m_\delta, +\infty]$), at most $r - m_\delta$ resources would be wasted where $r = (W_c - \sum_{j=1}^{i-1} t_j m_j) / \sum_{j=i}^n t_j + t_\delta$.*

Proof. According to Theorem 2, $x_{cv_\delta} = m_{nt}$ for any $m_{nt} \in (0, m_\delta]$ under the max-min fair allocation; this implies that the node would receive fewer resources if it reports fewer. If the node reports more resources, the central node may allocate more resources, but the resources beyond its true limitation would be wasted. From Lemma 11, the largest amount of resources that the new node could receive is $r = (W_c - \sum_{j=1}^{i-1} t_j m_j) / \sum_{j=i}^n t_j + t_\delta$; therefore, at most $r - m_\delta$ resources would be wasted. \square

Lemma 17. *When $R < t_\delta m_\delta$, there exists a real number r_2 satisfying the following conditions:*

- (i) *If $m_\delta \leq r_2$, for any $m_{nt} \in [0, m_\delta]$, x_{cv_δ} is a monotonically increasing function of m_{nt} and no resource is wasted; for any $m_{nt} \in (m_\delta, +\infty)$, at most $r_2 - m_\delta$ resources are wasted*
- (ii) *If $m_\delta > r_2$, for any $m_{nt} \in [0, m_\delta]$, $x_{cv_\delta} = m_{nt}$; for any $m_{nt} \in [r_2, +\infty)$, $x_{cv_\delta} = r_2 < m_\delta$; therefore, no resources are wasted*

Proof. To begin with, we derive the value of r_2 by the following three cases:

- (1) If $R \geq t_\delta m_n$, $r_2 = R/t_\delta$
- (2) If $m_1 (\sum_{j=1}^n t_j + t_\delta) \geq W_c$, $r_2 = W_c / \sum_{j=1}^n t_j + t_\delta$
- (3) If $R < t_\delta m_n$ and $m_1 (\sum_{j=1}^n t_j + t_\delta) < W_c$, from Lemma 11 to derive $r_2 = (W_c - \sum_{j=1}^{i-1} t_j m_j) / \sum_{j=i}^n t_j + t_\delta$

\square

\square

For simplicity, we only show the proof sketch of the third case. According to Lemma 11, when $m_{nt} \in [0, r_2]$, $x_{cv_\delta} = m_{nt}$, thus x_{cv_δ} is a monotonically increasing function of m_{nt} . Regardless of what value m_{nt} the new node reports, the received resources cannot be larger than r_2 . Hence, if $m_\delta \geq r_2$ and $m_{nt} \in [r_2, +\infty)$, $x_{cv_\delta} = r_2 < m_\delta$, and no resource is wasted. However, when $m_\delta \leq r_2$ and $m_{nt} \in [r_2, +\infty)$, the new node receives more resources than its true limitation, causing at most $r_2 - m_\delta$

resources wasted. Thus, the lemma holds. The proof of the other two cases are similar, and we omit the details.

According to Lemmas 16 and 17, the new node misreport that its resource limitation cannot gain more resources, and some resources would be wasted under some circumstances.

7. Truthfulness against the Spitting Strategy

The previous section shows that a node cannot gain more resources by misreporting its resource limitation. In this section, we propose a novel method which may help the node receive more resources. Suppose a new node's resource limitation is m_δ ; it reports to the central node that two nodes are inserted, and the sum of their resource limitations is m_δ . This implies that the node is assumed to be split into two nodes with resource limitations m_{np_1} , m_{np_2} , and $m_{np_1} + m_{np_2} = m_\delta$; we call this *splitting strategy*. We analyze when the node could achieve more resources through this method.

There are four cases according to the values of m_δ (the resource limitation of the new node), m_n (the largest resource limitation of the original nodes), and R (the spare resources before the new node is inserted, Equation (3)):

- (i) Case 1: $R \geq t_\delta m_\delta$
- (ii) Case 2: $R < t_\delta m_\delta$ and $R \geq 2t_\delta m_n$
- (iii) Case 3: $R < t_\delta m_\delta$ and $R < 2t_\delta m_n$

In this section, we show that the new node cannot gain more resources for the first two cases (Lemmas 18 and 19), but it can gain more resources for cases 3 (Lemma 20). Denote the two virtual nodes of v_δ as v_{np_1} , v_{np_2} and the received resources as $x_{cv_{np_1}}$, $x_{cv_{np_2}}$, respectively. Denote the received resources of node v_δ as x_{cv_δ} if it does not report two nodes. We show the following lemmas:

Lemma 18. *If $R \geq t_\delta m_\delta$, the splitting strategy has no effect to the new node, i.e., $x_{cv_{np_1}} + x_{cv_{np_2}} = x_{cv_\delta}$.*

Proof. Without loss of generality, suppose $m_{np_1} \leq m_{np_2}$ and $m_1 \leq m_2 \leq \dots \leq m_{k_1} \leq m_{np_1} \leq \dots \leq m_{np_2} \leq m_{k_2} \leq \dots \leq m_n$. When node v_{np_1} is inserted, we derive

$$\sum_{j=1}^{k_1} t_j m_j + m_{np_1} t_\delta + \sum_{j=k_1+1}^n t_j m_j < \sum_{j=1}^n t_j m_j + m_\delta t_\delta \leq W_c, \quad (21)$$

then $x_{cv_{np_1}} = m_{np_1}$. When node v_{np_2} is inserted, we derive

$$\begin{aligned} & \sum_{j=1}^{k_1} t_j m_j + m_{np_1} t_\delta + \sum_{j=k_1+1}^{k_2-1} t_j m_j + m_{np_2} t_\delta + \sum_{j=k_2}^n t_j m_j \\ & < \sum_{j=1}^n t_j m_j + (m_{np_1} + m_{np_2}) t_\delta \leq \sum_{j=1}^n t_j m_j + m_\delta t_\delta \leq W_c, \end{aligned} \quad (22)$$

then $x_{cv_{np_2}} = m_{np_2}$. Therefore, $x_{cv_{np_1}} + x_{cv_{np_2}} = m_{np_1} + m_{np_2} = m_\delta = x_{cv_\delta}$, which implies that the new node cannot gain more resources through the splitting strategy. \square

Lemma 19. *If $R < t_\delta m_\delta$ and $R \geq 2t_\delta m_n$, the splitting strategy has no effect to the new node, i.e., $x_{cv_{np_1}} + x_{cv_{np_2}} = x_{cv_\delta}$; thus, there is still no need to split the node.*

Proof. Since $R < t_\delta m_\delta$ and $R \geq 2t_\delta m_n$, we derive $t_\delta m_\delta > 2t_\delta m_n$, which implies $m_\delta > 2m_n$. Without loss of generality, suppose $m_{np_1} \leq m_{np_2}$, there are two cases

- (1) $m_{np_1} \leq m_n < m_{np_2}$ and $m_{np_1} + m_{np_2} = m_\delta > 2m_n$
- (2) $m_n < m_{np_1} \leq m_{np_2}$ and $m_{np_1} + m_{np_2} = m_\delta > 2m_n$

If $m_{np_1} \leq m_n < m_{np_2}$, suppose $m_1 \leq m_2 \leq \dots \leq m_{i-1} \leq m_{np_1} \leq m_i \leq \dots \leq m_n < m_{np_2}$; notice that

$$W_c \geq \sum_{j=1}^n t_j m_j + 2t_\delta m_n > \sum_{j=1}^{i-1} t_j m_j + t_\delta m_{np_1} + m_{np_1} \sum_{j=i-1}^n t_j. \quad (23)$$

When node v_{np_1} is inserted, $x_{cv_{np_1}} = m_{np_1}$. When node v_{np_2} is inserted, from $t_\delta m_\delta > W_c - \sum_{j=1}^n m_j t_j$, we have

$$t_\delta m_{np_2} > W_c - t_\delta m_{np_1} - \sum_{j=1}^n t_j m_j, \quad (24)$$

thus the allocated resources are

$$x_{cv_{np_2}} = \frac{W_c - t_\delta m_{np_1} - \sum_{j=1}^n t_j m_j}{t_\delta}. \quad (25)$$

Combining two nodes' resources, $x_{cv_{np_1}} + x_{cv_{np_2}} = m_{np_1} + ((W_c - t_\delta m_{np_1} - \sum_{j=1}^n t_j m_j)/t_\delta) = ((W_c - \sum_{j=1}^n t_j m_j)/t_\delta) = x_{cv_\delta}$.

For the second case, $m_n < m_{np_1} \leq m_{np_2}$ and $m_{np_1} + m_{np_2} > 2m_n$; notice that after the two nodes are inserted, for any node v_i , $i \in [1, n]$, $x_{cv_i} = m_i$. If $W_c - \sum_{j=1}^n t_j m_j \geq 2t_\delta m_{np_1}$, we derive

$$\begin{cases} x_{cv_{np_1}} = m_{np_1}, \\ x_{cv_{np_2}} = \frac{W_c - \sum_{j=1}^n t_j m_j - t_\delta m_{np_1}}{t_\delta}. \end{cases} \quad (26)$$

Therefore, $x_{cv_{np_1}} + x_{cv_{np_2}} = ((W_c - \sum_{j=1}^n t_j m_j)/t_\delta) = x_{cv_\delta}$. If $W_c - \sum_{j=1}^n t_j m_j \leq 2t_\delta m_{np_1}$, we derive $x_{cv_{np_1}} = x_{cv_{np_2}} = (W_c - \sum_{j=1}^n t_j m_j)/2t_\delta$, thus $x_{cv_{np_1}} + x_{cv_{np_2}} = ((W_c - \sum_{j=1}^n t_j m_j)/t_\delta) = x_{cv_\delta}$. Combining these cases, the lemma holds. \square

Lemma 20. *If $R < t_\delta m_\delta$ and $R < 2t_\delta m_n$, the new node can increase the received resources through the splitting method.*

For some special situations, the received resources could be about two times, i.e., $x_{cv_{np_1}} + x_{cv_{np_2}} \approx 2x_{cv_\delta}$.

Proof. If $R < t_\delta m_\delta$ and $R < 2t_\delta m_n$, there exists m_* such that the largest amount of resources two nodes can receive is $2m_*$. Suppose $m_1 \leq m_2 \leq \dots \leq m_{i-1} \leq m_* \leq m_i \leq \dots \leq m_n$; similar as Lemma 11, we derive

$$\sum_{j=1}^{i-2} t_j m_j + m_{i-1} \sum_{j=i-1}^n t_j + 2t_\delta m_{i-1} \leq W_c. \quad (27)$$

Thus, $m_{i-1} \leq (W_c - \sum_{j=1}^{i-2} t_j m_j) / (2t_\delta + \sum_{j=i-1}^n t_j)$, and we can find the largest value $i-1$ such that $m_{i-1} \leq m_*$ and $m_i \geq m_*$. Since $\sum_{j=1}^{i-1} t_j m_j + m_* \sum_{j=i}^n t_j + 2t_\delta m_* = W_c$, we derive

$$m_* = \frac{W_c - \sum_{j=1}^{i-1} t_j m_j}{2t_\delta + \sum_{j=i}^n t_j}. \quad (28)$$

According to Algorithm 1, if $m_* \leq m_{np_1} \leq m_{np_2}$, $x_{cv_{np_1}} = x_{cv_{np_2}} = m_*$. If $m_{np_1} \leq m_* \leq m_{np_2}$, we show that the sum of two nodes' resources is no more than $2m_*$.

Suppose $m_1 \leq m_2 \leq \dots \leq m_{i_1} \leq m_{np_1} \leq \dots \leq m_* \leq \dots \leq m_{i_2} \leq m_{np_2} \leq \dots \leq m_n$, it is easy to get $x_{cv_{np_1}} = m_{np_1}$ and $x_{cv_{np_2}} \geq m_*$. We analyze the maximum value $x_{cv_{np_2}}$ can achieve. Combining the inequality $\sum_{j=1}^{i_1} t_j m_j + t_\delta m_{np_1} + \sum_{j=i_1+1}^{i_2} t_j m_j + x_{cv_{np_2}} t_\delta + x_{cv_{np_2}} \sum_{j=i_2+1}^n t_j \leq W_c$ and the equation $\sum_{j=1}^{i_1-1} t_j m_j + m_* \sum_{j=i_1}^n t_j + 2t_\delta m_* = W_c$, we derive

$$\begin{aligned} (m_{np_1} + x_{cv_{np_2}} - 2m_*) t_\delta &\leq \left(\sum_{j=1}^{i_1-1} t_j m_j + m_* \sum_{j=i_1}^n t_j \right) \\ &\quad - \left(\sum_{j=1}^{i_2} t_j m_j + x_{cv_{np_2}} \sum_{j=i_2+1}^n t_j \right) \\ &\leq 0. \end{aligned} \quad (29)$$

Hence, $x_{cv_{np_1}} + x_{cv_{np_2}} \leq m_{np_1} + x_{cv_{np_2}} \leq 2m_*$. We then analyze when the $x_{cv_{np_1}} + x_{cv_{np_2}} = 2m_*$.

If $m_{np_2} > m_*$, $i \leq i_2$ and $(m_{np_1} + x_{cv_{np_2}} - 2m_*) t_\delta < 0$ (Equation (29)), the sum of two nodes' resources cannot be $2m_*$. When $x_{cv_{np_2}} = m_*$, we can derive $x_{cv_{np_1}} + x_{cv_{np_2}} = m_{np_1} + x_{cv_{np_2}} = 2m_*$. If the node is not split, the received resources is $x_{cv_\delta} = (W_c - \sum_{j=1}^{i-1} t_j) / (t_\delta + \sum_{j=i}^n t_j)$. Since $m_* = (W_c - \sum_{j=1}^{i-1} t_j m_j) / (2t_\delta + \sum_{j=i}^n t_j)$ (see Equation (28)), if $t_\delta < \sum_{j=1}^n t_j$, we derive $i_1 = i$, and the received resources of two split nodes are $2m_* \approx 2x_{cv_\delta}$, which implies that the node can receive about 2 times resources compared to the original allocation. Therefore, the lemma holds. \square

From the lemma, we know that the new node can achieve more resources by the splitting strategy. For some special situations, the node can gain appropriate two times resources compared to a honest node. Combining Lemmas 18, 19, and 20, we conclude the theorem as follows:

If the spare resources are adequate ($R \geq t_\delta m_\delta$) or inadequate ($R < t_\delta m_\delta$) but suits $R \geq 2t_\delta m_n$, the splitting strategy has no effect on the new node. If the spare resources suit $R < t_\delta m_\delta$ and $R < 2t_\delta m_n$, the node can gain more resources ($x_{cv_{np_1}} + x_{cv_{np_2}} > x_{cv_\delta}$), and in some special situations, the resources can be about two times ($x_{cv_{np_1}} + x_{cv_{np_2}} \approx 2x_{cv_\delta}$).

8. Truthfulness against Splitting Strategy in Tree Graph

In the previous section, we show that a node can gain more resources by the splitting strategy in the centralized resource system. However, in the hierarchy, resources are passed down layer by layer. The node cannot obtain resources from the original node that owns the resources but receives resources from other adjacent nodes. In this section, we will study how the splitting behavior of the parent node affects the child nodes. Suppose a node v_r has several child nodes v_{p_i} , $i \in [1, k_1]$, the limitation of which is m_{p_i} . Node v_{p_i} has several child node v_{c_j} , $j \in [1, k_2]$. Node v_{p_θ} reports to its parent node v_r that two nodes $v_{p_\theta}^1$ and $v_{p_\theta}^2$ are inserted and the sum of their resource limitation is m_{p_i} . We analyze the influence of node v_{p_i} 's splitting strategy on its child node v_{c_j} 's received resources. x_{ij} is the resources node v_i allocates to node v_j . W_r is the resources of node v_r . Let R_r be the spare resources after node v_r allocates its resources to its child node v_{c_i} except node v_{c_i} . We formulate it as $R_r = W_r - \sum_{i \in [1, p_\theta-1]} \cup_{[p_\theta+1, k_1]} x_{p_i} t_{p_i}$.

Lemma 21. *If the spare resources R_r are adequate ($R_r \geq t_{p_\theta} m_{p_\theta}$) or inadequate ($R_r < t_{p_\theta} m_{p_\theta}$) but suits $R \geq 2t_{p_\theta} m_{k_1}$, there are three cases:*

- (i) Case 1. $W_{p_\theta} > \sum_{i=1}^{k_2} m_{c_i} t_{c_i}$; there exists a splitting strategy for node v_{p_θ} to increase node v_{c_i} 's received resources
- (ii) Case 2. $W_{p_\theta} \leq m_{c_1} \sum_{i=1}^{k_2} t_{c_i}$; no matter how node v_{p_θ} splits, node v_{c_i} 's received resources do not change
- (iii) Case 3. $m_{c_1} \sum_{i=1}^{k_2} t_{c_i} < W_{p_\theta} \leq \sum_{i=1}^{k_2} m_{c_i} t_{c_i}$; there exists two real number r and r_1 satisfying the following:
 - (a) there exists a real number $r_1 = W_{p_\theta} / 2(\sum_{i=1}^{i_1-1} t_{c_i} m_{c_i} + m_{c_{i_1}} \sum_{i=i_1}^{k_2} t_{c_i})$ satisfying if $m_{c_i} \leq r_1$, v_{c_i} 's received resources increase and
 - (b) there exists a real number r such that if $m_{c_i} \geq r$, v_{c_i} 's received resources decrease

Proof. Suppose node v_{p_θ} reports two false node $v_{p_{\theta_1}}$ and $v_{p_{\theta_2}}$ to node v_r . Consider case 1 if $W_{p_\theta} \geq \sum_{i=1}^{k_2} m_{c_i} t_{c_i}$. Let $W_{p_{\theta_1}}$ and

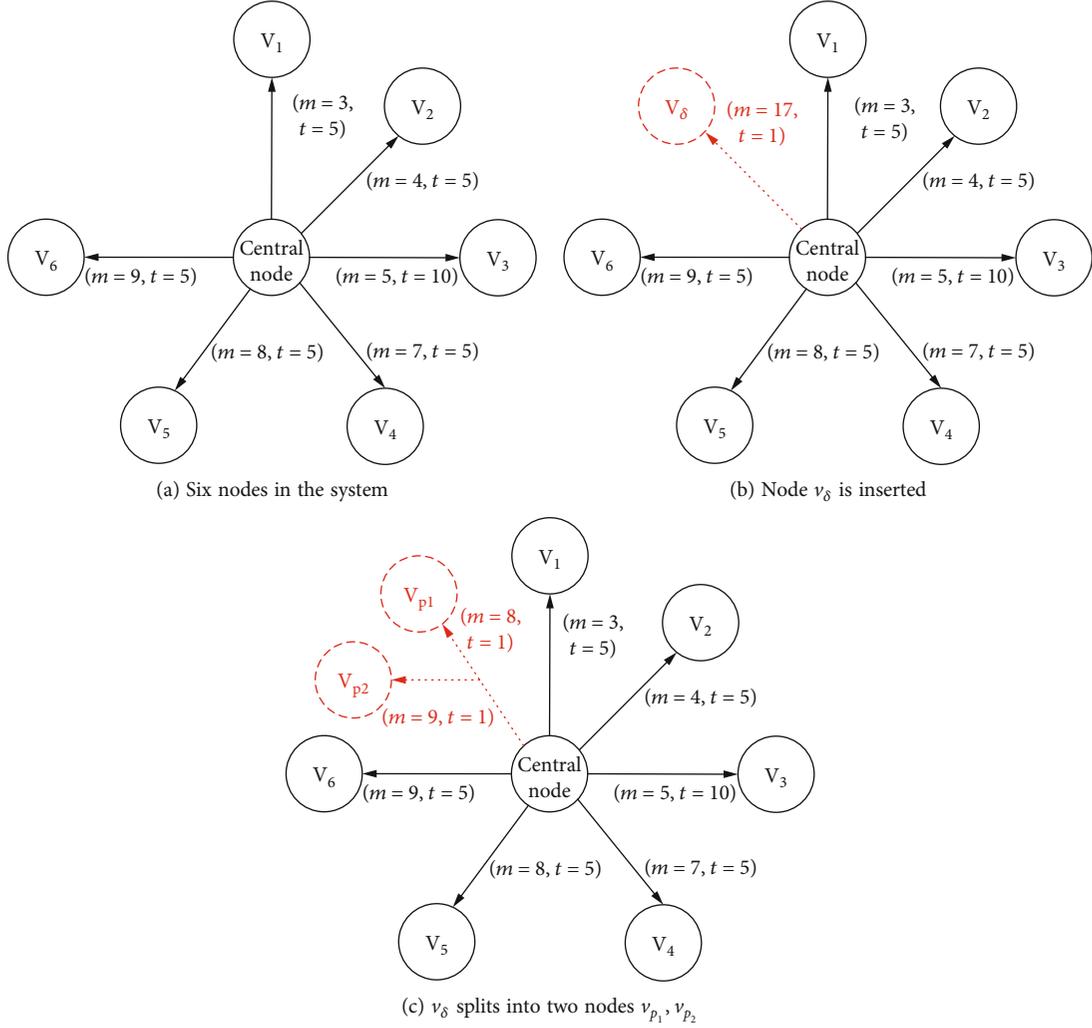


FIGURE 1: A centralized resource system: six nodes are connected to the central node; a new node v_δ is inserted with $m_\delta = 17$, $t_\delta = 1$; the node is split into two nodes v_{p1}, v_{p2} with $m_{p1} = 8$, $m_{p2} = 9$.

$W_{p_{\theta_2}}$ be the amount of resources two false nodes $v_{p_{\theta_1}}$ and $v_{p_{\theta_2}}$ receive from node v_r . If $W_{p_{\theta_1}} = \sum_{i=1}^{k_2} m_{t_{c_i}} t_{c_i}$ and $W_{p_{\theta_2}} = W_{p_\theta} - \sum_{i=1}^{k_2} m_{t_{c_i}} t_{c_i}$. Therefore, $x_{p_{\theta_1}} = m_{c_i}$, $x_{p_{\theta_2}} \geq 0$. $x_{p_{\theta_1}} + x_{p_{\theta_2}} \geq m_{c_i} = x_{p_\theta}$. v_{c_i} 's received resources increase. Consider case 2 if $W_{p_\theta} \leq m_{c_i} \sum_{i=1}^{k_2} t_{c_i}$, $x_{p_{\theta_1} c_i} = W_{p_{\theta_1}} / \sum_{i=1}^{k_2} t_{c_i}$, $x_{p_{\theta_2} c_i} = W_{p_{\theta_2}} / \sum_{i=1}^{k_2} t_{c_i}$, $x_{p_{\theta_1} c_i} = x_{p_{\theta_1} c_i} + x_{p_{\theta_2} c_i}$. Node v_{c_i} 's received resource does not change. Consider case 3 (a): suppose $W_{p_{\theta_1}} > W_{p_{\theta_2}}$, then $W_{p_{\theta_1}} > W_{p_\theta} / 2$ and $W_{p_{\theta_2}} < W_{p_\theta} / 2$. There must exist a real number $r_2 \geq r_1$ such that any node v_{c_i} , $i \in [1, k_2]$ satisfying $m_{c_i} \leq r_2$, $x_{p_{\theta_1}} = m_{c_i} = x_{p_\theta}$, and $x_{p_{\theta_2}} > 0$ under the new max-min fair allocation vector. $x_{p_{\theta_1}} + x_{p_{\theta_2}} > x_{p_\theta}$. v_{c_i} 's received resources increase. Consider case 3 (b): for node $v_{p_{\theta_1}}$'s allocation, there exists an integer $i_1^* \in [0, k_2]$ such that $x_{p_{\theta_1} c_i} = m_{c_i}$, $i \in [1, i_1^* - 1]$ and $x_{p_{\theta_1} c_i} < m_{c_i}$, $i \in [i_1^*, k_2]$. Similarly, for node $v_{p_{\theta_2}}$'s allocation, there exists an integer $i_2^* \in [0, k_2]$ such that $x_{p_{\theta_2} c_i} = m_{c_i}$, $i \in [1,$

$i_2^* - 1]$ and $x_{p_{\theta_2} c_i} < m_{c_i}$, $i \in [i_2^*, k_2]$. If $W_{p_{\theta_1}} > W_{p_{\theta_2}}$, then $i_1^* > i_2^*$. For node v_{p_θ} 's allocation, we derive

$$W_{p_\theta} - \sum_{j=1}^{i_1^*-1} t_j m_j = \sum_{j=i_1^*}^{i_2^*-1} t_j m_j + x_{p_{\theta_2} c_j} \sum_{j=i_2^*}^{k_2} t_j. \quad (30)$$

For false nodes $v_{p_{\theta_1}}$'s and $v_{p_{\theta_2}}$'s allocation, we derive

$$W_{p_{\theta_1}} - \sum_{j=1}^{i_1^*-1} t_j m_j = x_{p_{\theta_1} c_j} \sum_{j=i_1^*}^{k_2} t_j, \quad (31)$$

$$W_{p_{\theta_2}} - \sum_{j=1}^{i_2^*-1} t_j m_j - x_{p_{\theta_2} c_j} \sum_{j=i_2^*}^{i_1^*-1} t_j = x_{p_{\theta_2} c_j} \sum_{j=i_1^*}^{k_2} t_j. \quad (32)$$

Combining Equation (30), Equation (31), and Equation (32), we derive

$$\left(x_{p_{\theta_1 c_j}} + x_{p_{\theta_2 c_j}}\right) \sum_{j=i_1^*}^{k_2} t_j < \sum_{j=i_1^*}^{i^*-1} t_j m_j + x_{p_{\theta c_j}} \sum_{j=i^*}^{k_2} t_j. \quad (33)$$

Therefore, $(x_{p_{\theta_1 c_j}} + x_{p_{\theta_2 c_j}}) < x_{p_{\theta c_j}}$; otherwise, we derive $\sum_{j=i_1^*}^{i^*-1} t_j m_j + x_{p_{\theta c_j}} \sum_{j=i^*}^{k_2} t_j \leq x_{p_{\theta c_j}} \sum_{j=i_1^*}^{k_2} t_j \leq (x_{p_{\theta_1 c_j}} + x_{p_{\theta_2 c_j}}) \sum_{j=i_1^*}^{k_2} t_j$, which makes a contradiction. Thus, the lemma holds. \square \square

By a similar method, the following lemma can be derived.

Lemma 22. *If the spare resources suit $R_r < t_{p_0} m_{p_0}$ and $R_r < 2t_{p_0} m_{k_1}$, there are three cases:*

- (i) *Case 1. $W_{p_0} > \sum_{i=1}^{k_2} m_{c_i} t_{c_i}$ or $W_{p_0} \leq m_{c_i} \sum_{i=1}^{k_2} t_{c_i}$; node v_{c_i} can gain more resources by v_{p_0} 's splitting strategy*
- (ii) *Case 2. $m_{c_i} \sum_{i=1}^{k_2} t_{c_i} < W_{p_0} \leq \sum_{i=1}^{k_2} m_{c_i} t_{c_i}$; there exists two real number r and r_1 satisfying (a) if $m_{c_i} \geq r$, v_{c_i} 's received resources decrease ($t_{p_0} > \sum_{j \in [1, p_0-1] \cup [p_0+1, k_1]} t_j$) or increase ($t_{p_0} < \sum_{j \in [1, p_0-1] \cup [p_0+1, k_1]} t_j$), and (b) there exists a real number $r_1 = W_{p_0} / 2(\sum_{i=1}^{i_1-1} t_{c_i} m_{c_i} + m_{c_{i_1}} \sum_{i=i_1}^{k_2} t_{c_i})$ satisfying: if $m_{c_i} \leq r_1$, v_{c_i} 's received resources increase*

From Lemmas 21 and 22, we know that if grandparent node's resources are adequate, node's resources can not increase by parent's splitting strategy in most scenarios. However, if grandparent node's resources are inadequate, in most cases, node's resources can increase by parent's splitting strategy.

9. Numerical Examples

In this section, we present some numerical examples to illustrate these results. As shown in Figure 1(a), six nodes are connected to the central node in the system, and (m, t) (resource limitation, transfer rate) is depicted on each edge. For example, node v_1 has $m_1 = 3, t_1 = 5$. Assume the amount of all resources the central node holds is $W_c = 205$.

By Algorithm 1, the max-min allocation is listed in Table 2 where u_i is the utility of the node which is equal to the received resources x_{cv_i} ; $x_{cv_i} \cdot t_i$ is the resources the central node allocates. There is no spare resource since $R = W_c - \sum_{i=1}^6 x_{cv_i} t_i = 0$. As shown in Figure 1(b), if a new node v_δ is inserted with $m_\delta = 17, t_\delta = 1$, the max-min allocation is presented as Table 3. Nodes v_5, v_6 receive fewer resources since their resources are allocated to the new node. If the node misreports its limitation m_δ , it cannot gain more resources, and some resources would be wasted.

TABLE 2: The max-min fair allocation of six nodes.

Node i	m_i	t_i	x_{cv_i}	$x_{cv_i} \cdot t_i$	u_i
v_1	3	5	3	15	3
v_2	4	5	4	20	4
v_3	5	10	5	50	5
v_4	7	5	7	35	7
v_5	8	5	8	40	8
v_6	9	5	9	45	9

TABLE 3: The max-min allocation after v_δ is inserted.

Node i	m_i	t_i	x_{cv_i}	$x_{cv_i} \cdot t_i$	u_i
v_1	3	5	3	15	3
v_2	4	5	4	20	4
v_3	5	10	5	50	5
v_4	7	5	7	35	7
v_5	8	5	$85/11 \approx 7.727$	$425/11$	$85/11$
v_6	9	5	$85/11 \approx 7.727$	$425/11$	$85/11$
v_δ	17	1	$85/11 \approx 7.727$	$85/11$	$85/11$

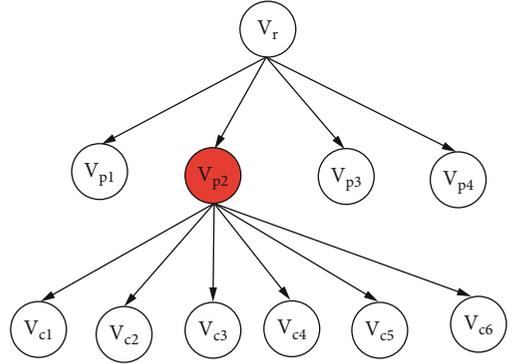


FIGURE 2: A tree with root v_r , whose resources are passed down to descendant nodes.

In order to gain more resources, the new node splits itself into two nodes v_{p_1}, v_{p_2} such that $m_{p_1} = 8, m_{p_2} = 9$ ($m_{p_1} + m_{p_2} = m_\delta = 17$) (as shown in Figure 1(c)). The max-min allocation is presented in Table 1; nodes v_5, v_6 receive fewer resources, and nodes v_{p_1}, v_{p_2} both receive $85/12$ resources. Therefore, with the splitting strategy, the new node receives resources $x_{cv_{p_1}} + x_{cv_{p_2}} = 85/6 \approx 14.166$. Compared to the received resources in Table 3, the splitting strategy achieves $14.166/7.727 \approx 1.83$ times that of the original allocation.

Next, we present an example to illustrate splitting strategy in tree graph.

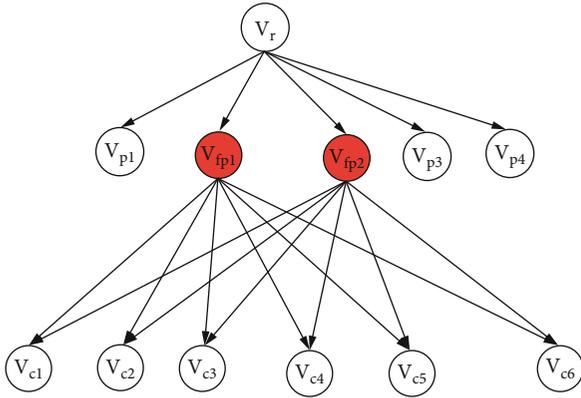
The overall amount of node v_r is $W_r = 100$. As shown in Figure 2, node v_r allocates its resources to its child $v_{p_i}, i \in [1, 4]$. The resource demand m and resource transfer rate t are shown in Table 4. The amount of resource node v_{p_2} 's

TABLE 4: The max-min allocation of node v_r 's resources.

Node i	m_{pi}	t_{pi}	x_{rpi}
v_{p1}	12	2	12
v_{p2}	8	2	8
v_{p3}	20	1	20
v_{p4}	24	1	24

TABLE 5: The max-min allocation of node v_{p2} 's resources.

Node i	m_{ci}	t_{ci}	x_{p2ci}
v_{c1}	1	1	1
v_{c2}	2	3	2
v_{c3}	4	2	$17/5 \approx 3.4$
v_{c4}	5	1	$17/5 \approx 3.4$
v_{c5}	6	1	$17/5 \approx 3.4$
v_{c6}	8	1	$17/5 \approx 3.4$

FIGURE 3: A tree with two false nodes v_{fp1} and v_{fp2} whose resources are passed down to descendent nodes.TABLE 6: The max-min allocation of two false nodes v_{fp1} and v_{fp2} 's resources.

Node i	m_{ci}	t_{ci}	x_{fp1ci}	x_{fp2ci}
v_{c1}	1	1	1	1
v_{c2}	2	3	$11/8 \approx 1.373$	$11/8 \approx 1.373$
v_{c3}	4	2	$11/8 \approx 1.373$	$11/8 \approx 1.373$
v_{c4}	5	1	$11/8 \approx 1.373$	$11/8 \approx 1.373$
v_{c5}	6	1	$11/8 \approx 1.373$	$11/8 \approx 1.373$
v_{c6}	8	1	$11/8 \approx 1.373$	$11/8 \approx 1.373$

receives is 24. Once receiving the resources from node v_r , node v_{p2} will allocate its resources to its child v_{ci} , $i \in [1, 6]$. The max-min allocation is presented as Table 5. The amount of resource node v_{ci} , $i \in [3, 6]$ receives is fewer than their resource demand. When node v_{p2} splits itself into two false

node v_{fp1} and v_{fp2} (as shown in Figure 3), the overall amount of resources node v_{ci} , $i \in [3, 6]$ receives decreases. For instance, the amount of resource node v_{p2} allocates to node v_{c3} is $x_{p2c3} = 17/5 \approx 3.4$. However, the overall resource node v_{c3} receives from two false nodes v_{fp1} and v_{fp2} equals $x_{fp1c3} + x_{fp2c3} = 22/8 \approx 2.75 < 3.4$ (as presented in Table 6).

10. Conclusion

In this paper, we study the truthfulness of the max-min fair allocation in a centralized resource system. Max-min fairness is a common adopted concept in evaluating an allocation mechanism. We propose a simple but efficient algorithm to find out the allocation. On the foundation of the algorithm, we analyze how the allocation varies when a new node is added to the system. Furthermore, we explore whether the allocation is truthful against two strategic behaviors. The misreporting strategy cannot affect the allocation while the new node can gain more resources by the splitting strategy. Hence, the max-min allocation mechanism is not truthful especially against the splitting strategy. Finally, we analyze the effect of node's splitting strategy on its child nodes on tree graph. In some situations, nodes can gain more resources despite the splitting strategy of its parents. It is still an open problem to design a fair and truthful allocation mechanism in a centralized resource system.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

The paper is an extended version of our WASA 2020 paper "Can the Max-Min Fair Allocation Be Trustful in a Centralized Resource System?"

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported in part by the Guangdong Province Key Research and Development Plan under Grant No. 2019B010136003, the National Key Research and Development Program of China (2019QY1406), the National Natural Science Foundation of China under Grant No. 61972106, the Guangdong Higher Education Innovation Group (2020KCXTD007), and the Guangzhou Higher Education Innovation Group (202032854).

References

- [1] Steinhaus, "The problem of fair division," *Econometrica*, vol. 16, pp. 101–104, 1948.
- [2] G. Gamow and M. Stern, *Puzzle-Math*, the Viking Press, New York, NY, USA, 1958.

- [3] S. J. Brams and A. D. Taylor, "Fair division: from cake-cutting to dispute resolution," *Social Justice Research*, vol. 12, no. 2, pp. 149–162, 1999.
- [4] Y. Chen, J. K. Lai, D. C. Parkes, and A. D. Procaccia, "Truth, justice, and cake cutting," *Games & Economic Behavior*, vol. 77, no. 1, pp. 284–297, 2013.
- [5] V. Menon, K. Larson, V. Menon, K. Larson, V. Menon, and K. Larson, *Deterministic, Strategyproof, and Fair Cake Cutting*, IJCAI, 2017.
- [6] S. Wahlen and M. Laamanen, *Collaborative Consumption and Sharing Economies*, Routledge, 2017.
- [7] L. Hurwicz, "On Informationally Decentralized Systems," *Decision and Organization: A volume in Honor of Jacob Marschak*, pp. 297–336, 1972.
- [8] B. Adsul, C. S. Babu, J. Garg, R. Mehta, and M. Sohoni, *Nash Equilibria in Fisher Market*, Springer, Berlin Heidelberg, 2010.
- [9] N. Chen, X. Deng, H. Zhang, and J. Zhang, "Incentive ratios of fish markets," in *International Colloquium Conference on Automata, Languages, and Programming*, pp. 464–475, Warwick, UK, 2012.
- [10] Y. Cheng, X. Deng, Y. Pi, and X. Yan, *Can Bandwidth Sharing be Truthful?*, Springer, Berlin Heidelberg, 2015.
- [11] Y. Cheng, X. Deng, Q. Qi, and X. Yan, "Truthfulness of a proportional sharing mechanism in resource exchange," in *International Joint Conference on Artificial Intelligence (IJCAI)*, pp. 187–193, NY, USA, 2016.
- [12] E. L. Hahne, "Round-robin scheduling for max-min fairness in data networks," *IEEE Journal on Selected Areas in Communications*, vol. 9, no. 7, pp. 1024–1039, 1991.
- [13] A. Charny, *An Algorithm for Rate Allocation in a Packet*, Massachusetts Institute of Technology, 1994.
- [14] P. Cretch, J. Michielsens, and J. Taeymans, *Data Network*, Prentice Hall, 2002.
- [15] Z. Chen, Y. Cheng, X. Deng, Q. Qi, and X. Yan, "Agent incentives of strategic behavior in resource exchange," in *International Symposium on Algorithmic Game Theory*, pp. 227–239, L'Aquila, Italy, 2017.
- [16] Y. Sun, Y. Yu, and J. Han, "Ranking-based clustering of heterogeneous information networks with star network schema," in *Proceedings of the 15th ACM SIGKDD international conference on Knowledge discovery and data mining - KDD '09*, pp. 797–806, Paris, France, 2009.
- [17] Y. Sun, J. Han, P. Zhao, Z. Yin, H. Cheng, and T. Wu, "Rankclus: integrating clustering with ranking for heterogeneous information network analysis," in *Proceedings of the 12th International Conference on Extending Database Technology Advances in Database Technology - EDBT '09*, pp. 565–576, Saint Petersburg, Russia, 2009.
- [18] B. Becker and H. U. Simon, "How robust is the n-cube?," *Information and Computation*, vol. 77, no. 2, pp. 162–178, 1986.
- [19] E. Segal-Halevi, *Cutting a Cake with Both Good and Bad Parts*, 2017.
- [20] E. Segal-Halevi and S. Nitzan, "Envy-Free Cake-Cutting among Families," 2016, <http://arxiv.org/abs/1607.01517>.
- [21] L. Chen, S. Liu, B. Li, and B. Li, "Scheduling jobs across geodistributed datacenters with max-min fairness," *IEEE Transactions on Network Science and Engineering*, vol. 6, no. 3, pp. 488–500, 2019.
- [22] Y. Jia, M. Zhao, and W. Zhou, "Joint user association and eicic for max-min fairness in hetnets," *IEEE Communications Letters*, vol. 20, no. 3, pp. 546–549, 2016.
- [23] J. Khamse-Ashari, G. Kesidis, I. Lambadaris, B. Urgaonkar, and Y. Zhao, "Constrained max-min fair scheduling of variable-length packetows to multiple servers," *Annals of Telecommunications*, vol. 4, pp. 1–19, 2018.
- [24] T. V. Pham and A. T. Pham, "Max-min fairness and sum-rate maximization of mu-vllocal networks," in *2015 IEEE Globecom Workshops (GC Wkshps)*, pp. 1–6, San Diego, CA, USA, 2015.
- [25] C. Aloui, B. Hkiri, S. Hammoudeh, and M. Shahbaz, "A multiple and partial wavelet analysis of the oil price, inflation, exchange rate, and economic growth nexus in saudi arabia," *Emerging Markets Finance & Trade*, vol. 54, no. 4, pp. 935–956, 2018.
- [26] N. S. Ghumman and R. Kaur, "Dynamic combination of improved max-min and ant colony algorithm for load balancing in cloud system," in *2015 6th International Conference on Computing, Communication and Networking Technologies (ICCCNT)*, pp. 1–5, TX, USA, July 2015.
- [27] M. Blot, M. Cord, and N. Thome, "Max-min convolutional neural networks for imageclassification," in *2016 IEEE International Conference on Image Processing (ICIP)*, pp. 3678–3682, Phoenix, AZ, USA, September 2016.
- [28] M. Feldman, K. Lai, I. Stoica, and J. Chuang, "Robust incentive techniques for peer-to-peer networks," in *ACM Conference on Electronic Commerce*, pp. 102–111, New York, NY, USA, May 2004.
- [29] N. Chen, X. Deng, and J. Zhang, "How protable are strategic behaviors in a market?," in *Algorithms - ESA 2011. ESA 2011. Lecture Notes in Computer Science, vol 6942*, C. Demetrescu and M. M. Halldórsson, Eds., pp. 106–118, Springer, Berlin, Heidelberg, 2011.
- [30] T. Roughgarden and E. Tardos, "How bad is selfish routing?," *Journal of the ACM*, vol. 49, no. 2, pp. 236–259, 2002.
- [31] D. Bertsekas and R. Gallager, *Data Networks*, Prentice-Hall Inc., 2nd edition, 1992.
- [32] B. Radunovic and J. Y. le Boudec, "A Unified framework for max-min and min-max fairness with applications," *IEEE/ACM Transactions on Networking*, vol. 15, no. 5, pp. 1073–1083, 2007.